Periodic and quasi-periodic orbits in Celestial Mechanics models with dissipation

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PLANETARY MOTIONS, SATELLITE DYNAMICS, AND SPACESHIP ORBITS

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Outline

Conformally symplectic systems
  Quasi-Periodic solutions in a map

Analyticity Breakdown Transition

Sobolev method
  A KAM Theorem
  Proof and fast computations
  Sobolev method

Greene’s method
  A partial justification
  Idea of the proof

Numerical Examples
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Numerical Examples
Conformally symplectic systems transport a symplectic form into a multiple of itself

- **Gaussian thermostat**
  
  Appear in non-equilibrium statistical mechanics (e.g. Wojtowski and Liverani).

- **Spin-Orbit models**
  
  Appear in celestial mechanics (e.g. Biasco, Chierchia, and Celletti).

- **Any Hamiltonian system with friction proportional to the velocity**

- **Any two dimensional diffeomorphism**
Conformally symplectic flow

Let $\Omega$ be a symplectic form such that

$$\Omega_x(u, v) = (u, J(x)v)$$

and $X$ a vector field such that there exists a function $\eta : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathcal{L}_X \Omega = \eta \Omega.$$

The time $t$ flow $f_t$ satisfies that

$$(f_t)^* \Omega = \exp(\eta t) \Omega.$$
Let $\Omega$ be a symplectic form such that

$$\Omega_x(u, v) = (u, J(x)v).$$

A conformally symplectic map $f : T^n \times \mathbb{R}^n \to T^n \times \mathbb{R}^n$ is

$$f^* \Omega = \lambda \Omega$$

for $\lambda \in \mathbb{R}$. 
Dissipative spin-orbit model

The spin-orbit equation with tidal torque

\[ \ddot{x} + \varepsilon \left( \frac{a}{r} \right) \sin(2x - 2f) = -\eta e (\dot{x} - \nu e) \]
For a given resonance of order $p : 2$ we define the resonant angle $\gamma = x - \frac{p}{2} t$ and averaging the conservative contribution over the orbital period

$$\ddot{\gamma} + \varepsilon W \left( \frac{p}{2}, e \right) \sin(\gamma) = -\eta_e (\dot{\gamma} - \nu_e)$$

or

$$\dot{\gamma} = \Gamma$$

$$\dot{\Gamma} = -\varepsilon W \left( \frac{p}{2}, e \right) \sin(2\gamma) - \eta_e (\Gamma - \nu_e)$$
A conformally symplectic map

\[
\begin{align*}
\dot{\gamma} &= \Gamma \\
\dot{\Gamma} &= -\varepsilon W \left( \frac{p}{2}, e \right) \sin(2\gamma) - \eta_e (\Gamma - \nu_e)
\end{align*}
\]

Euler’s symplectic method gives a map

\[
\begin{align*}
\Gamma' &= \Gamma + h \left[ -\varepsilon W \left( \frac{p}{2}, e \right) \sin(2\gamma) - \eta_e (\Gamma - \nu_e) \right] \\
\gamma' &= \gamma + h\Gamma' \\
t' &= t + h
\end{align*}
\]

Which is a conformally symplectic map

\[
\begin{align*}
\Gamma_{n+1} &= \lambda \Gamma_n + \mu + \varepsilon V'(\gamma_n) \\
\gamma_{n+1} &= \gamma_n + \Gamma_{n+1}
\end{align*}
\]
Q-P solutions in models with dissipation

Conformally symplectic systems

\[ \omega = \frac{p}{q} \quad \text{or} \quad \omega \neq \frac{p}{q}, \quad p \in \mathbb{Z}, \ q \in \mathbb{N} \]
Dissipative standard map

The map $f_{\mu}$ given by

\[
\begin{align*}
    y_{n+1} &= \lambda y_n + \mu + \varepsilon \, V'(x_n) \\
    x_{n+1} &= x_n + y_{n+1}
\end{align*}
\]

- $|Df_{\mu}| = \lambda$
- $\lambda = 1, \mu = 0$ -standard map.
- $0 < \lambda < 1$ dissipative map.
Dissipative standard map for $\varepsilon = 0$

\[\omega = \frac{\mu}{1 - \lambda},\]

\[\omega = \frac{p}{q},\]

$\omega \neq \frac{p}{q}$. 
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Numerical Examples
Fix a Diophantine frequency

1. KAM theory ensures the existence of smooth quasi-periodic solutions for “quasi-integrable” system
2. There are examples with no smooth equilibria
3. Breakdown on quasi-periodic attractors is also related to the size of the basin of attraction
Where is the boundary of existence of smooth solutions?

What happens near the boundary?
The transition is very similar to the phase transition and there are scaling relations and Renormalization Group transformations.

There are extensive numerical studies in twist mappings, some numerical studies in the case of non-twist mappings, and much less in extended systems or conformally symplectic systems.

Some aspects of the Renormalization Group have been proved rigorously.

Renormalization Group theory for conformally symplectic systems (Rand)
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Numerical Examples
Quasi-periodic solutions are orbits of the form

\[(q_n, p_n) = K(n\omega), \quad \omega \in \mathbb{R} \setminus \mathbb{Q}\]

In such a case, we have

\[f_{\mu} \circ K(\theta) = K(\theta + \omega).\]

We will assume

\[K(\theta + 1) = K(\theta) + (1, 0)\]

“non-contractible circles”.

We fix a Diophantine frequency $\omega$

$$|\omega \cdot k - n| \geq \nu |k|^{-\tau}, \quad \forall k \in \mathbb{Z} \setminus \{0\}, n \in \mathbb{N}.$$ 

The invariance equation for $K_0 \in \mathcal{A}_\rho$, $\mu_0 \in \Lambda$

$$Err(\theta) = f_{\mu_0} \circ K_0(\theta) - K_0(\theta + \omega)$$

A solution is $K_e$, $\mu_e$ so that $f_{\mu_e} \circ K_e(\theta) = K_e(\theta + \omega)$. 
We will need to evaluate expressions that are explicit in $K_0$

\[
T_0 = \left\| \begin{pmatrix} \bar{S} & \frac{S(B_b)^0 + (M_0^{-1} \circ T_\omega D_\mu f_\mu \circ K_0)_1}{(M_0^{-1} \circ T_\omega D_\mu f_\mu \circ K_0)_2} \\ (\lambda - 1) \text{Id} & -1 \end{pmatrix} \right\|^{-1}
\]

with $\lambda(B_b)^0(\theta) - (B_b)^0(\theta + \omega) = -(M_0^{-1} \circ T_\omega D_\mu f_\mu \circ K_0)_2$

\[
M_0(\theta) = (DK_0(\theta) | J^{-1} DK_0(\theta) N_0(\theta))
\]

\[
N_0(\theta) = (DK_0(\theta)^T DK_0(\theta))^{-1}
\]
Spaces of Sobolev functions

Given $m > 0$ and denoting the Fourier series of a function $f = f(z)$ as $f(z) = \sum_{k \in \mathbb{Z}^n} \hat{f}_k \exp(2\pi i k z)$, we define the space $H^m$ as

$$H^m = \left\{ f : \mathbb{T}^n \rightarrow \mathbb{C} : \| f \|_m \equiv \left( \sum_{k \in \mathbb{Z}^n} | \hat{f}_k |^2 (1 + |k|^2)^m \right)^{1/2} < \infty \right\}.$$

- Suitable for finite differentiable problems
- These spaces are very computable if we are working with Fourier Coefficients.
Q-P solutions in models with dissipation

- Sobolev method
- A KAM Theorem

Finite regularity

Theorem (C-Celletti-de la Llave)

- \( f_\mu \in C^r, \ r > m + 13\tau, \ m > n/2 + \tau \)
- \( K_0 \in H^{m+13\tau}, \ \mu_0 \in \Lambda, \ \lambda \neq 1 \)
- \( \| Err \|_{H^{m-\tau}} \leq C(\nu, \tau, n, T_0, \| DK_0 \|_{A_0}, \| M_0^{\pm 1} \|_{A_\rho}, \| N_0 \|_{A_\rho}) \)

Then \( \exists \ K_e \in H^{m-\tau}, \ \mu_e \in \Lambda \) such that
\( f_{\mu_e} \circ K_e(\theta) - K_e(\theta + \omega) = 0 \) \text{ and } \( K_0 - K_e \|_{H^{m-\tau}} \leq C \| Err \|_{H^m}, \)

\( |\mu_0 - \mu_e| \leq C \| Err \|_{H^m}. \)
Consequences of \textit{a posteriori} theorems

- Validation of numerical algorithms producing a solution
- Continuation methods of numerical analysis
- Local uniqueness and bootstrap of regularity results
- $\implies$ A numerically accessible criterion for breakdown of analyticity (and hyperbolicity)

\textit{Note: we do not assume that the system is close to integrable}
The proof of the existence theorem is based on a Nash–Moser Newton–like method.

- $Err(\theta) = E[K, \mu](\theta)$ is small,
- A “better” solution $(\tilde{K}, \tilde{\mu})$ satisfies
  $$E[\tilde{K}, \tilde{\mu}] = E + DE[K, \mu](\tilde{K} - K) + D_{\mu}E[K, \mu](\tilde{\mu} - \mu) + O(\| (\tilde{K}, \tilde{\mu}) - (K, \mu) \|^2),$$
- $\tilde{K} = K - DE^{-1}[K, \mu](E + D_{\mu}E[K, \mu] \sigma)$
- $\tilde{\mu} = \mu + \sigma$
- $\| E[\tilde{K}, \tilde{\mu}] \| \approx \| Err \|^2.$

By means of Nash–Moser theory we prove the convergence of the algorithm.
We assume that

$$Err(\theta) = E[K, \mu](\theta) = f_\mu \circ K(\theta) - K(\theta + \omega)$$

is small for $K$ and $\mu$.

Then the corrections $\Delta$ and $\sigma$ satisfy

$$\tilde{K} = K + \Delta$$

$$\tilde{\mu} = \mu + \sigma$$

and

$$Df_\mu(K(\theta))\Delta(\theta) - \Delta(\theta + \omega) + \partial_\mu f_\mu(\theta)\sigma = -Err(\theta).$$

Nash–Moser theory allows to use an approximate inverse in place of an inverse.
A new frame of reference

\[ Df_{\mu}(K(\theta))\Delta(\theta) - \Delta(\theta + \omega) + \partial_{\mu} f_{\mu}(\theta)\sigma = -E(\theta) \]

\[ DE(\theta) = Df_{\mu}(K(\theta))DK(\theta) - DK(\theta + \omega) \]

\[ V(\theta) = J^{-1}DK(\theta)N(\theta) \]

\[ V(\theta + \omega) = J^{-1}DK(\theta + \omega)N(\theta + \omega) \]
New frame of reference

\[ Df_\mu(K(\theta))\Delta(\theta) - \Delta(\theta + \omega) + \partial_\mu f_\mu(\theta)\sigma = -E(\theta) \]

- Picture for 1-dimension
- In higher dimensions this is just a computation
New frame of reference

If $M(\theta) = (DK(\theta)|V(\theta))$ then

$$Df_\mu(K(\theta))M(\theta) = M(\theta + \omega) \begin{pmatrix} I_n & S(\theta) \\ 0 & \lambda I_n \end{pmatrix}.$$ 

In particular, if

$$\Delta(\theta) = M(\theta)W(\theta)$$

we have that, $O(N^3)$ operations,

$$Df_\mu(K(\theta))\Delta(\theta) - \Delta(\theta + \omega) + \partial_\mu f_\mu(\theta)\sigma = -Err(\theta)$$

reduces to, $O(N \log(N))$ operations,

$$W_1(\theta) - W_1(\theta + \omega) = -(M^{-1}(\theta + \omega)[Err(\theta) + \partial_\mu f_\mu(\theta)\sigma])_1 + S(\theta)W_2(\theta),$$

$$\lambda W_2(\theta) - W_2(\theta + \omega) = -(M^{-1}(\theta + \omega)[Err(\theta) + \partial_\mu f_\mu(\theta)\sigma])_2.$$
The computation is implementable

The Quasi-Newton Step consists in using geometric identities to find an approximate solution of the linearized equation using only

- Shifting functions
- Multiplying and composing functions
- Differentiating functions
- Solving difference equations with constant coefficients

The same geometric cancellations above can be used to obtain fast and stable numerical methods.

\[ O(N \log N) \]
The reduction works in any dimension. Note that we can solve for $W$ from

$$W_1(\theta) - W_1(\theta + \omega) = - (M^{-1}(\theta + \omega)[Err(\theta) + \partial_\mu f_\mu(\theta)\sigma])_1 + S(\theta)W_2(\theta),$$

$$\lambda W_2(\theta) - W_2(\theta + \omega) = - (M^{-1}(\theta + \omega)[Err(\theta) + \partial_\mu f_\mu(\theta)\sigma])_2.$$

The reduction works both for $\lambda \neq 1$ and $\lambda = 1$.

There are analogous reductions for flows.
Local uniqueness and bootstrap of regularity are given. In practice, the functionals we need to check are:

- Non-degeneracy of the problem
- That the approximate solution is rather regular
Choose a path in the parameter space starting in the integrable case.

**Initialize** the parameters and the solution at integrable

**Repeat**

- Increase the parameters along the path
- Run the iterative step

  **If** (Iterations do not converge)
  - Decrease the increment in parameters

  **Else** (Iteration success)
  - Record the values of the parameters and the Sobolev norm of the solution.

**Until** Sobolev norm too large

Then, we know the breakdown occurs.
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Numerical Examples
Residue for an area preserving map $f : \mathbb{R} \times \mathbb{T} \to \mathbb{R} \times \mathbb{T}$ with a $p/q$-periodic orbit

$$R = \frac{1}{4}(2 - Tr(Df^q))$$

where $Df^q$ is computed along the full cycle of the orbit.

Invariant circle exists if and only if $R$ is bounded for $p/q$-orbit approximating $\omega$. 
Partial justifications for symplectic case

Falcolini, de la Llave, Mackay

- Compute the Birkhoff normal form up to high enough order around the invariant circle
- Error in approximate integrability can be bounded and residue is invariant under changes of variables
- Obtain upper bounds for the residues of the periodic orbits approximating torus

All periodic orbits with rotation number close to $\omega$ will have small residue
An analog of the residue

Let $\mathcal{M}$ be a symplectic manifold of dimension $2n$ and $f : \mathcal{M} \to \mathcal{M}$ a conformally symplectic map with a point whose orbit has rotation number $\rho = \left(\frac{a_1, \ldots, a_n}{L}\right)$. Consider

$$c(x) = x^{2n} + c_{2n-1}x^{2n-1} + \cdots + c_1x + \lambda^{nL}$$

the characteristic polynomial of the derivative over a full cycle of the periodic orbit with frequency $\rho$. Then if

$$(x - 1)^n(x - \lambda^L)^n = x^{2n} + c_{2n-1}^0x^{2n-1} + \cdots + c_1^0x + \lambda^{nL},$$

we define the residue

$$R = \sum_{j=1}^{2n-1} |c_j - c_j^0|.$$
A linearization theorem

The main advantage of conformally symplectic systems

**Theorem (C-Celletti-de la Llave)**

*If $f$ is analytic and $K : \mathbb{T}^n \to \mathcal{M}$ is an analytic embedding then there exists an analytic diffeomorphism $g$ from $\mathcal{M}_\rho$ to a neighborhood of $K(\mathbb{T}^n)$ such that*

$$g^{-1} \circ f \circ g(A, \theta) = (\lambda A, \theta + \omega)$$

$$g^*\Omega_0 = \Omega$$

$$g(\theta, 0) = K(\theta)$$
The spectrum has a pairing rule $\text{Spec}(Df^L) = \{\gamma_i, \lambda^L\gamma_i^{-1}\}$

**Theorem (C-Falcolini-Celletti-de la Llave)**

$f_\mu \in C^r, \mu \in \mathbb{R}^\ell$ conformally symplectic s.t. $f_0$ admits a Lagrangian invariant torus with rotation $\omega$.

Then $\exists \mathcal{U}$ neighborhood of the torus such that whenever $(\text{orbit of } x) \in \mathcal{U}$ then $\exists C_{N,r,\tau} > 0$ with

$$|\gamma_i - 1| \leq LC_N \|\mu\|^N$$

We use this estimate to find bounds on the spectral numbers of the periodic orbit
Partial justification

We are looking for bounds on spectral number of an orbit of type $\rho$

$$|\gamma_i - 1| \leq L C_N \|\mu\|^N$$

- $\|\mu\|$ is comparable to $\|\omega - \rho\|$ (non-degeneracy condition)
- There is a sequence of $L$ that almost saturates

$$L \approx \|\omega - \rho\|^{-\frac{1}{(\tau+1)}}$$

- Then for $N$ sufficiently large, $|\gamma_i - 1| \rightarrow 0$
- The problem of approximating vectors by rational vectors remains
Deformation theory

- A smooth change of variables that reduces the system to $(\phi + S_\mu, \lambda l)$ up to an error (Normal form)
- The spectrum is invariant under smooth changes of variables
- For systems in normal form the residue is zero and the spectral numbers are 1.
- Estimate the spectrum by bounding the error in the normal form using the theory of deformations (de la Llave, Banyaga, Wayne, Marco, Moriyón)
NHIM and averaging

- NHIM theory (Fenichel, Hirsch, Pugh, Shub) → $T_\mu$ is a family of invariant tori under $f_\mu$
- $R_\mu(\phi)$ the dynamics of $f_\mu$ on $T_\mu$
- Averaging theory tells us that we can find $B_\mu^N$

$$\left(B_\mu^N\right)^{-1} \circ R_\mu \circ B_\mu^N = T_{\omega_\mu^N} + O(\|\mu\|^{N+1})$$

- Periodic orbits have $n$ Lyapunov exponents close to 1
- Pairing rule and Lagrangian character of the tori $\Rightarrow$ remaining exponents are close to $\lambda$
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Numerical Examples
Example: the dissipative standard map

\( f_\mu \) is given by

\[
\begin{align*}
y_{n+1} &= \lambda y_n + \mu + \varepsilon V'(x_n) \\
x_{n+1} &= x_n + y_{n+1}
\end{align*}
\]

\( Err \equiv f_\mu \circ K - K \circ T_\omega \)

\( \tilde{K} = K + \Delta, \quad \tilde{\mu} = \mu + \delta \)

\[
Df_\mu \circ K \Delta - \Delta \circ T_\omega + \delta \partial_\mu f_\mu = -Err
\]

\( O(N^3) \)
Example: the dissipative standard map

$f_\mu$ is given by

\[ y_{n+1} = \lambda y_n + \mu + \varepsilon V'(x_n) \]
\[ x_{n+1} = x_n + y_{n+1} \]

\[ \text{Err} \equiv f_\mu \circ K - K \circ T_\omega \]

\[ \tilde{K} = K + M W, \]
\[ \tilde{\mu} = \mu + \delta \]

\[ W_1(\theta) - W_1(\theta + \omega) = \eta_1(\theta, \sigma) - S(\theta) W_2(\theta), \]
\[ \lambda W_2(\theta) - W_2(\theta + \omega) = \eta_2(\theta, \sigma). \]

\[ O(N \log N) \]
Joint work with A. Celletti and J-L Figueras

\[ K(\theta) = \begin{pmatrix} \theta + u(\theta) \\ \omega + u(\theta + \omega) - u(\theta) \end{pmatrix} \]

Figure: Invariant circle and conjugacy \( V'(x) = \frac{\varepsilon}{2\pi} \sin(2\pi x) \) for \( \varepsilon = 0.97176356, \lambda = 0.9 \)

\( \omega \) is the golden mean
Figure: Existence domain. Parameter space \((\varepsilon_1, \varepsilon_2)\). The potential 
\[ V'(x) = \frac{\varepsilon_1}{2\pi} \sin(2\pi x) + \frac{\varepsilon_2}{4\pi} \sin(4\pi x) \] for \(\lambda = 1.0\)
Figure: Existence domain. Parameter space $(\varepsilon_1, \varepsilon_2)$. The potential $V'(x) = \frac{\varepsilon_1}{2\pi} \sin(2\pi x) + \frac{\varepsilon_2}{4\pi} \sin(4\pi x)$ for $\lambda = 0.9$
Figure: Existence domain. Parameter space $(\varepsilon_1, \varepsilon_2)$. The potential $V'(x) = \frac{\varepsilon_1}{2\pi} \sin(2\pi x) + \frac{\varepsilon_2}{4\pi} \sin(4\pi x)$ for $\lambda = 0.5$
Figure: Existence domain. Parameter space \((\varepsilon_1, \varepsilon_2)\). The potential \(V'(x) = \frac{\varepsilon_1}{2\pi} \sin(2\pi x) + \frac{\varepsilon_2}{4\pi} \sin(4\pi x)\) for \(\lambda = 0.1\).
Figure: Existence domain. Parameter space $(\varepsilon_1, \varepsilon_2)$. The potential $V'(x) = \frac{\varepsilon_1}{2\pi} \sin(2\pi x) + \frac{\varepsilon_2}{4\pi} \sin(4\pi x)$ for $\lambda = 0.01$
Reducibility of the cocycle: joint work with Figueras

\[ y_{n+1} = \lambda y_n + \mu + \varepsilon V'(x_n), \]
\[ x_{n+1} = x_n + y_{n+1}. \]

We can compute the reduction
\[
Df_\mu(K(\theta)) \tilde{M}(\theta) = \tilde{M}(\theta + \omega) \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}.
\]

Then, the center and stable bundles are
\[
\tilde{M}(\theta) = [W^c(\theta) | W^s(\theta)]
\]
and we can compute the minimum angle.
Figure: Stable and center bundle for $\lambda = 0.4$ and $\varepsilon = 0.5$
A strange non-chaotic attractor

Figure: Stable and center bundle $\lambda = 0.4$ and $\varepsilon = 0.98057339$
Celletti and Chierchia (2009)
For the differential equation,

\[ \ddot{x} + \eta(\dot{x} - \nu) + \varepsilon V(x, t) = 0, \]

the quasi-periodic attractor is parameterized by

\[ T = \left\{ (x, t) = (\theta_1 + u(\theta_1, \theta_2), \theta_2); \theta \in T^2 \right\}. \]

Let

\[ \partial_\omega = \omega \frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2}, \]

then the conjugacy \( u \) satisfies the PDE

\[ \partial_\omega^2 u + \eta \partial_\omega u + \varepsilon V(\theta_1 + u(\theta_1, \theta_2), \theta_2) + \beta = 0 \]

with \( \beta = \eta(\omega - \nu) \) when \( \varepsilon = 0. \)
The constructive proof suggests an efficient numerical algorithm.

Figure: Conjugacy $u(\theta_1, \theta_2)$ for $\eta = 0.25$, $e = 0.0001$, $\varepsilon = 16.60780498$
**Figure:** Blow-up curve and Fourier spectrum of conjugacy for 
\( \eta = 0.25, \ e = 0.0001, \ \varepsilon = 16.60780498 \)
Summary:

- The breakdown of analyticity in conformally symplectic systems is very relevant to models in Celestial Mechanics.
- A-posteriori formalism provides:
  1. very efficient numerical algorithms
  2. rigorous justification for a Sobolev norm method to detect the breakdown of analyticity
  3. a partial justification of Greene’s criterion
  4. rigorous justification for Greene’s criterion
- These techniques suggest many extensions (work in progress)
Thank you
### Q-P solutions in models with dissipation

#### Numerical Examples

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| Rigorous justification               | ⇒  | ⇒  | ⇔   | ⇔   | ⇔   | ⇒  | ⇔      |

| Requires periodic orbits             | Y  | Y  | Y   | Y   | N   | N  | N      |

| Interpretation of the Renor. Group   | Y  | Y  | N   | ?   | ?   | Y  | ?      |

| Requires Positive Def.               | N  | N  | Y   | Y   | Y   | Y  | N      |
Joint work with J. Ll. Figueras

**Figure:** Spectrum of the conjugacy $u$ and invariant circle $K$ computed with $N \approx 10^6$ Fourier modes for $\lambda = 0.4$. 