The aspect of the planetary normal form

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Overview

- A ``sun'' with mass $\bar{m}_0$
- N ``planets'' with masses $m\,\bar{m}_1$, $\cdots$, $m\,\bar{m}_N$, where $m \ll 1$.

undergoing gravitational interaction.

- In a Poincaré variables, the Hamiltonian of the system has the form

$$H_P = h(\Lambda_1, \cdots, \Lambda_N) + m f_P(\underbrace{\Lambda_1, \cdots, \Lambda_N}_{R^N}, \underbrace{l_1, \cdots, l_N}_{T^N}, \underbrace{u_1, \cdots, u_{2N}, v_1, \cdots, v_{2N}}_{R^{2N}})$$

  where h, f are related to sun-planet, planet-planet interaction.

- The integrable part $h(\Lambda_1, \cdots, \Lambda_N)$ looses degrees of freedom (proper degeneracy).
- The ``secular'' (averaged) perturbation

$$(f_P)_{av}(\Lambda, u, v) = \frac{1}{(2pi)^N} \int_{T^N} f_P(\Lambda, l, u, v) dl$$

  has and elliptic equilibrium point at (u,v)=0, but the first order Birkhoff invariants are identically resonant.

- Planetary Birkhoff Normal Form: to conjugate $H_{(1+N)B}$ to a new system $H_{BNF} = h + m f_{BNF}$, where the average

$$(f_{BNF})_{av} := \frac{1}{(2pi)^N}\int_{T^N} f_{BNF}$$

has the form

$$(f_{BNF})_{av} = P_p\Big(\frac{u_1^2+v_1^2}{2}, \cdots, \frac{u_{2N}^2+v_{2N}^2}{2}; \Lambda\Big) + O((u,v)^{2p+1})$$

where $P_p(J;\Lambda)$ is a polynomial of degree p in $J = (J_1, \cdots, J_{2N})$.

- BNF has been obtained in [Chierchia-P., Invent.Math., 2011] (based on [P. PhD Thesis, 2009]) by foliating the 6N--dimensional phase space into symplectic, invariant, (6N-2)--dimensional leaves parametrized by the ‘‘direction of the total angular momentum C’’. On any of such invariant manifold an explicit, global set of symplectic coordinates has been obtained such that the system takes the same form, independently of the leaf, and degeneracies do not appear.

Consequences of BNF:

- direct proof of Arnold's planetary theorem and estimate of the measure of the ‘‘Kolmogorov set’’ [P., PhD Thesis 2009]; [Chierchia-P. Invent.Math., 2011];
- proof of Nekhorossev-like ‘‘full stability’’ of eccentricities and inclinations over polynomially-long times [Chierchia-P. Journ.Mod.Dyn., 2011], excluding mean-motion resonances;
- estimating the measure of the closure of periodic orbits in NBP [Chierchia, Proc. ICMP., 2012];
- ‘‘full stability’’ of eccentricities over exponentially-long times in the planar 3BP, excluding mean-motion resonances [P. 2013, preprint] ;
- proof of independence of the density of the ‘‘Kolmogorov set’’ of eccentricities and mutual inclinations in the spatial 3BP and planar NBP [P. 2013, preprint].

Other perspectives of application:

- to prove exponential ‘‘full stability’’ in the general problem, away from mean-motion resonances;
- to study how the BNF changes in presence of mean motion resonances; for example, study of the resonance 1:1 in the three--body problem [P., work in progress];
- to prove instability around some mean motion resonance. Results in this direction have been obtained by [Féjoz, Guárdia, Kaloshin, Roldàn, preprint 2011] for ER3BP.

V. I. Arnold's Planetary Theorem

Theorem [V. I. Arnold, 1963] ''In the many-body problem there exists a set of initial conditions having a positive Lebesgue measure and such that, if the initial positions and velocities belong to this set, the distances of the bodies from each other will remain perpetually bounded, provided the masses of the planets are sufficiently small.''

- Arnold proves APT in the case of the planar 3BP;
- He then provides a sketchy proof for the spatial 3BP and only some arguments for the general (1+N)BP;
- In the 90's M. R, Herman realizes that
  - the arguments given in [Arnold 1963] for the spatial three--body case are wrong;
  - the general case $(1+N) \geq 3$ is not proven in [Arnold 1963] (it relies on a conjecture).
- A great work follows from the French Celestial Mechanics school (Abdullah, Albouy, Chenciner, Féjoz, Herman, Laskar, Malige, Robutel, ...), around APT, to try to give it a complete proof.

N. N. Nekhorossev's Stability of Semi--Axes

Theorem [N. N. Nekhorossev, 1977] Let

$$H = h(I) + m\, f(I, x, u, v) \qquad m \ll 1$$

$\exists$ a, b, $m_0 > 0$ s.t. if

(i) H is real--analytic for $(I, x, u, v) \in A^{n_1} \times T^{n_1} \times B^{2n_2}$, $T := R/(2piZ)$

(ii) $m < m_0$

(iii) h is ''steep'' (in particular, convex)

(iv) $(I(t), x(t), u(t), v(t))$ is a solution s.t. $(u(t), v(t)) \in B^{2n_2}$ for all $0 \le t \le T_\star := T_0 e^{m^{-a}}$

then

$$|I_i(t) - I_i(0)| \le R_\star := R_0 m^b \qquad \text{for} \qquad |t| \le T_\star \ .$$

- Nekhorossev easily applies his theorem to the planetary Hamiltonian H_P written in Poincaré variables and infers stability of semi--major axes over exponentially--long times.
- He leaves the problem of stability of eccentricities and inclinations completely open.

The Classical Hamiltonian setting

(1) The Heliocentric Reduction (3(1+n)→3n d.o.f.)

$$H_{(1+N)BP} = \underbrace{\sum_{i=1}^{N}\left(\frac{|y^{(i)}|^2}{2m_i} - \frac{m_i M_i}{|x^{(i)}|}\right)}_{\substack{h_{Kep} \\ \text{(integrable)}}} + \underbrace{m \sum_{1\le i<j\le N}\left(\frac{y^{(i)}\cdot y^{(j)}}{\bar{m}_0} - \frac{\bar{m}_i\bar{m}_j}{|x^{(i)} - x^{(j)}|}\right)}_{\substack{f_{(1+N)BP} \\ \text{(perturbing function)}}}$$

$x^{(i)},\ y^{(i)} \in R^3\ ,\quad x^{(i)} \neq 0\ ,\quad x^{(i)} \neq x^{(j)}$

$\bar{m}_0$: mass of the ''star''

$m\,\bar{m}_i\ (m \ll 1\ ,\quad i = 1, 2, \cdots, N)$: masses of the ''planets'' ,

$M_i = \bar{m}_0 + m\bar{m}_i \quad m_i = \dfrac{\bar{m}_0\bar{m}_i}{\bar{m}_0 + m\bar{m}_{i\,i}}$ ''reduced'' masses .

(2) Delaunay coordinates (3n d.o.f.)

Fix a orthonormal triple $(k^{(1)}, k^{(2)}, k^{(3)}) \in (R^3)^3$; define ''node'' n_i:=intersection of the i^{th} orbital plane and the plane $(k^{(1)}, k^{(2)})$, $i = 1$, ..., N.

$$\begin{cases} \Lambda_i := M_i \sqrt{m_i a_i} \\ l_i := \text{mean anomaly of } x^{(i)} \text{ on } E_i \end{cases}$$

$$\begin{cases} G_i := \left|C^{(i)}\right| \\ g_i := \text{anomaly of the Perihelion } P_i \\ \qquad \text{w.r.t. } n_i \end{cases}$$

$$\begin{cases} T_i := C^{(i)} \cdot k^{(3)} \\ t_i := \text{anomaly of } n_i \text{ w.r.t. } k^{(1)} \end{cases}$$

(3) Poincaré coordinates

Poincaré coordinates:

$$\begin{cases} \Lambda_i := \Lambda_i \\ \bar{l}_i := l_i + g_i + t_i \qquad \text{mean longitude} \end{cases}$$

$$\begin{cases} \eta_i = \sqrt{2(\Lambda_i - G_i)} \cos(l_i + g_i) \\ \xi_i = -\sqrt{2(\Lambda_i - G_i)} \sin(l_i + g_i) \end{cases}$$

$$\begin{cases} p_i := \sqrt{2(G_i - T_i)} \cos t_i \\ q_i = -\sqrt{2(G_i - T_i)} \sin t_i \end{cases}$$

The Planetary Hamiltonian in Poincaré coordinates:

$$H_P = \underbrace{-\sum_{i=1}^{N} \frac{M_i^2 m_i^3}{2\Lambda_i^2}}_{\text{N d.o.f.}} + m \underbrace{f_P(\Lambda, \bar{l}, z)}_{\text{3N d.o.f.}} \qquad z = (z_1, \cdots, z_N) \ , \quad z_i = (\eta_i, \xi_i, p_i, q_i)$$

V. I. Arnold's ''Fundamental Theorem'' (1963)

Theorem [V. I. Arnold, Usp. Math. Nauk. 1963] Consider a Hamiltonian of the form $H(I,x,u,v) = H_0(I) + mP(I,x,u,v)$ which is real--analytic on $P_{\bar{e}_0} := V \times \mathbb{T}^{n_1} \times B^{2n_2}_{\bar{e}_0}$. Assume that

(i) $I \in V \to D_I H_0$ is a diffeomorphism;

(ii) P_{av} is in Birkhoff normal form of order 6;

(iii) the matrix T(I) (''torsion'') of the coefficients of degree 2 is non--singular on V.

Then, there exists $\bar{e}_0$, C such that, if $e < \bar{e}_0$ and $m < e^8$, one can find a set $K_{m,e}$ such that its Lebeasgue measure verifies

$$\text{meas}\, K_{m,e} \geq (1 - Ce^a)\, \text{meas}\, P_e \quad a = \frac{1}{16(n_1+n_2)}$$

of , analytic, Lagrangian, (n_1+n_2)--dimensional tori where the motion is conjugated to the linear flow $x \in T^{n_1+n_2} \to x + ot$ with irrational frequencies $o \in R^{n_1+n_2}$.

Degeneracies in the secular problem

- Consider the system in Poincaré variables $H_P = h_{Kep} + mf_P$.
- Up to a diagonalization of its quadratic part, the averaged perturbation is

$$(f_P)_{av} = \sum_{i=1}^{N} (s_i \frac{\eta_i^2 + \xi_i^2}{2} + s_i' \frac{p_i^2 + q_i^2}{2}) + O(\eta, \xi, p, q)^4$$

- The following linear combinations with integer coefficients hold

$$s_N' \equiv 0 \qquad \sum_{i=1}^{N} (s_i + s_i') \equiv 0 \quad (\text{``secular degeneracies''})$$

- The two resonances above were the main reason of M. R. Herman's criticism to Arnold's approach to the proof of APT, since the BNF required in FT was in serious doubt.

The Proof of Arnold's Planetary Theorem (1963-2011)

- [Arnold 1963]: planar 3BP; checks assumptions of FT;
- [Robutel 1995]: spatial 3BP; reduces the SO(3)--symmetry via ''Jacobi reduction''; checks assumptions of FT of the reduced system;
- [Herman 2009]: Computes torsion of planar (1+N)BP (unpublished); as for the spatial problem, changes strategy w.r.t. Arnold's project;
- [Fejoz 2004]: general NBP, uses Herman's (weaker) KAM Theory and indirect arguments to bypass degeneracies;
- [Chierchia-P.2011]: Reduction of degeneracies and use of (a refined) FT. Measure estimates of the ''Kolmogorov Set''. Result based on [P. 2009; PhD Thesis]

Chierchia-P. 2011's statement of APT

Theorem A [Chierchia-P., Invent. Math, 2011]
Consider (1+N) point masses: a ''star'' $\bar{m}_0$ and N planets $m\bar{m}_1$ ..., $m\bar{m}_N$ interacting through gravity. There exist positive numbers $\bar{e}_0 < 1 < C$ and $\underline{a}_i < \bar{a}_i < \underline{a}_{i+1}$ $(i = 1, \cdots, N)$ such that, for any

$$e < \bar{e}_0 \ , \quad m < e^3/C$$

in the phase space

$$P_e : \ \underline{a}_i \le a_i \le \bar{a}_i \ , \quad \max_{i,j}\{e_i, i_j\} \le e$$

one can find a set $K_{m,e}$ (''Kolmogorov set'') such that its Lebeasgue measure verifies

$$\text{meas}\, K_{m,e} \ge (1 - C\sqrt{e})\, \text{meas}\, P_e$$

of analytic, Lagrangian, (3N-1)--dimensional tori where the motion is conjugated to the linear flow $x \in T^{3N-1} \to x + ot$ with irrational frequencies $o \in R^{3N-1}$.

Theorem A is the main result of my PhD Thesis (Roma Tre, 2009). For this result, I am indebted to L. Chierchia, my advisor.

‘‘Full’’ stability over polynomially-long times, away from mean-motion resonances

Theorem B [Chierchia-P., Journ. Mod. Dyn., 2011] Fix $0 < c < 1$, $p \in \mathbb{N}$. There exist $\bar{e}_0 < 1 < C$ and $\underline{a}_i < \bar{a}_i < \underline{a}_{i+1}$ and an open subset of

$$\underline{a}_i \le a_i \le \bar{a}_i \ , \quad \bar{e}_{min} \le \max_{i,j}\{e_i,\ i_j\} \le \bar{e}_{max} < \bar{e}_0 \ , \quad i_n \ge \bar{e}_{max} - \frac{\bar{e}_{min}}{4}$$

then

$$\max_{i,j}\ |e_i(t) - e_i(0)|, \qquad |i_j(t) - i_j(0)| \le c\bar{e}_{min} \quad \text{for} \quad 0 \le t \le \frac{C}{\bar{e}^p_{min}}$$

Full stability over exponentially--long times in the planar three--body problem

Theorem C [P. 2013] Let N=2, planar. There exist $\bar{e}_0 < 1 < C$, $\underline{a}_i < \bar{a}_i < \underline{a}_{i+1}$ and an open subset of

$$\underline{a}_i \le a_i \le \bar{a}_i \quad , \quad \bar{e}_{min} \le \max\{e_1,\ e_2\} \le \bar{e}_{max} < \bar{e}_0 \quad , \quad e_2 \ge \bar{e}_{max} - \frac{\bar{e}_{min}}{4}$$

(free of mean motion resonances of sufficiently high order) such that, for all initial data in such subset,

$$\max_{i=1,2} |e_i(t) - e_i(0)| \le d^b \quad \text{for} \quad 0 \le t \le e^{d^{-a}}$$

with $d := \max\{m,\ e_{min}\}$.

The Planar Three--Body Case [Arnold 1963]

‘‘ In the case of three bodies [on a plane] we can to obtain stronger results than those of 3. if conservation of angular momentum (see §5) is used. It turns out that it is not necessary to require the eccentricities to be small; all that is necessary is that they should be small enough to exclude the possibility of collision.’’

- In the planar three--body case case, $(f_P)_{av}$ is integrable
- a less general result (a simplified version) than FT can be applied.
- By such simpler result, Arnold states the following improved result

$$\text{meas}\, K_{m,\bar{e}_0} \geq (1 - C\, m^b)\, \text{meas}\, P_{\bar{e}_0} \qquad (*)$$

uniformly in e_0. Note that

- FT is not really necessary for the planar three--body case.
- Due to the flaw in [Arnold 1963], an analogue uniform bound as in (*) for the spatial three--body case is in doubt.

Proving (part of) Arnold's conjecture for the Spatial Three--Body Problem and the Planar One

Theorem D [P. 2013] (i) Let N=2, spatial.

$$\mathrm{meas}\, K_{m,\bar{e}_0} \geq \left(1 - C\, m^b - C(\frac{a_1}{a_2})^b\right) \mathrm{meas}\, P_{\bar{e}_0}$$

uniformly in e_0.

(ii) The same result holds for the planar NBP.

On the proof of Theorem A

Symplectic Reduction by Rotations [P. 2008; PhD]

Fix

- a orthonormal triple $F_0 = (k^{(1)}, k^{(2)}, k^{(3)}) \in (\mathbb{R}^3)^3$;
- auxiliary frames: $F_1, \cdots, F_{N-1}$;
- orbital frames: $F_1^*, \cdots, F_N^* \quad (k_{*i}^{(3)} \parallel C^{(i)})$
- partial sums: $S^{(j+1)} := \sum_{p=1}^{j+1} C^{(p)}$

Define $(\Lambda, \Gamma, \Psi, l, \gamma, y) \in \mathbb{R}^{3N} \times \mathbb{T}^{3N}$ via

$$\begin{cases} \Psi_N = C_3 = C \cdot k^{(3)} \\ \Psi_{N-1} = G_0 = |C| \\ \Psi_j = |S^{(j+1)}| \end{cases} \qquad \begin{cases} \psi_N = z & \text{longitude of } C \text{ w.r.t. } F_0 \\ \psi_{N-1} = g & \text{longitude of } C^{(N)} \text{ w.r.t. } F_1 \\ \psi_j = & \text{longitude of } C^{(j+1)} \text{ w.r.t. } F_{N-j} \end{cases}$$

$$\begin{cases} \Gamma_i = |C^{(i)}| \\ \gamma_i & \text{longitude of } P_i \text{ w.r.t. } F_i^* \end{cases} \qquad \begin{cases} \Lambda_i \\ l_i \end{cases} = \text{Delaunay's}$$

- The variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ extend to the general case $N \geq 2$ the well known ‘‘Jacobi reduction of the nodes’’, 1842, available for N=2.
- In 2008, I thought to have discovered the variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ during my PhD, that I completed in 2009 in Rome, University Roma Tre, under the direction of L. Chierchia.
- I then realized that the $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ were symplectically related to a different set of symplectic variables previously studied by [A. Deprit, 1983], who in turn developed ideas by [F. Boigey, 1982], after reading the paper [Malige, Robutel, Laskar, 2002], where Deprit’s paper was mentioned;
- For this reason, the variables $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$ were attributed to A. Deprit and named ‘‘Deprit variables’’;
- The proof of the symplectic character of the $(\Lambda, \Gamma, \Psi, \ell, \gamma, \psi)$’s found in [P. PhD Thesis, 2009] and their relation with the original Deprit’s set was then published in [Chierchia-P. ‘‘Deprit’s reduction of the nodes revisited’’, Celest. Mech.2011].

Regularization of Eccentricities and Inclinations

[P.'s PhD Thesis, 2009]; [Chierchia-P. Invent.Math., 2011]

$$\begin{cases} p_N = \sqrt{2(|C| - C_3)}\cos z \\ q_N = -\sqrt{2(|C| - C_3)}\sin z \end{cases} \quad \text{Integrals}$$

$$\left.\begin{cases} \Lambda_i \\ \bar{\bar{l}}_i = l_i + \gamma_i + \psi^N_{i-1} \\ \eta_i = \sqrt{2(\Lambda_i - \Gamma_i)}\cos\left(\gamma_i + \psi^N_{i-1}\right) \\ \xi_i = -\sqrt{2(\Lambda_i - \Gamma_i)}\sin\left(\gamma_i + \psi^N_{i-1}\right) \\ p_i = \sqrt{2(\Gamma_{i+1} + \Psi_{i-1} - \Psi_i)}\cos\psi^N_i \\ q_i = -\sqrt{2(\Gamma_{i+1} + \Psi_{i-1} - \Psi_i)}\sin\psi^N_i \end{cases} \quad \psi^N_i := \sum_{j=i}^{N} \psi_j \ , \quad \psi_0 = 0 \right\} \quad M^{6N-2} = \{(p_N, q_N) = \text{fixed}\}$$

The system has an extra--integral

$$G_0 = |C| = \sum_{i=1}^N \Lambda_i - \sum_{i=1}^N \frac{\eta_i^2 + \xi_i^2}{2} - \sum_{i=1}^{N-1} \frac{p_i^2 + q_i^2}{2}$$

The Partially Reduced System (Proof of Theorem A)

$$H_{pr} = h_{Kep}(\Lambda) + m f_{pr}(\Lambda, \bar{\bar{l}}, \bar{z}) \quad \bar{z} = (\eta_1, \cdots, \eta_n, p_1, \cdots, p_{N-1}, \xi_1, \cdots, \xi_N, q_1, \cdots, q_{N-1})$$

$$(f_{pr})_{av} \sim C_0 + \sum_{i=1}^{N} (s_i \frac{\eta_i^2 + \xi_i^2}{2} + \sum_{i=1}^{N-1} s_i' \frac{p_i^2 + q_i^2}{2} + O(\eta, \xi, p, q)^4$$

- The s_i, s_j' are the same as in Poincaré setting, except for $s_N' \equiv 0$;
- They may be computed (asymptotically). They do not verify resonances other than Herman's for well separated semi--major axises. This improves a result by [J. Féjoz, 2004], who proves that no resonances are identically satisfied by s_i, s_j' other than the secular resonances;
- The system exhibits D'Alembert rules; in particular, there exists an extra--symmetry, caused by the integral G_0.
- The normal form exists up to any desired order 2p; Herman resonance does not affect the construction of the normal form. Non--trivial torsion is checked by induction on N.

The Aspect of the Planetary Torsion (1)

[P.'s PhD Thesis, 2009]; [Chierchia-P. Invent.Math., 2011]

N=2

$$\mathrm{T} = \mathrm{T}_2 \quad = \quad m_1 m_2 \frac{a_1^2}{a_2^3} \begin{pmatrix} \frac{3}{4\Lambda_1^2} & -\frac{9}{4\Lambda_1\Lambda_2} & \frac{3}{\Lambda_1^2} \\ -\frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{\Lambda_2^2} & \frac{9}{4\Lambda_1\Lambda_2} \\ \frac{3}{\Lambda_1^2} & \frac{9}{4\Lambda_1\Lambda_2} & -\frac{3}{4\Lambda_1^2} \end{pmatrix} \left(1 + O\left(\frac{\Lambda_1}{\Lambda_2}\right)\right)$$

The Aspect of the Planetary Torsion (2)

$N \geq 3$, $3 \leq k \leq N$

$$T = \begin{pmatrix} T_2 & & & \\ & T_3 & & \\ & & \ddots & \\ & & & T_N \end{pmatrix} \qquad T_k = \bar{m}_{k-1}\bar{m}_k \frac{a_{k-1}^2}{a_k^3} \begin{pmatrix} -\frac{3}{\Lambda_k^2} & \frac{9}{4\Lambda_{k-1}\Lambda_k} \\ \frac{9}{4\Lambda_{k-1}\Lambda_k} & -\frac{3}{4\Lambda_{k-1}^2} \end{pmatrix}$$

- T is non--singular: this proves Theorem A;
- T is not sign--definite (its eigenvalues have alternating sign);
- The integrable truncation of the planetary system

$$h_{\mathrm{Trunc}} = h_{\mathrm{Kep}}(\Lambda) + m(C_0(\Lambda) + (s(\Lambda), s'(\Lambda)) \cdot J + \frac{1}{2} J \cdot T(\Lambda) J + \cdots)$$

is not convex (nor quasi--convex).

Full Reduction of SO(3)-Symmetry

The remaining integral $G_0 = |C| = \sum_{i=1}^{N} \Lambda_i - \sum_{i=1}^{N} \frac{\eta_i^2+\xi_i^2}{2} - \sum_{i=1}^{N-1} \frac{p_i^2+q_i^2}{2}$ may be eliminated letting

$$\begin{cases} \Lambda_j = \Lambda_j \\ \bar{\bar{l}}_j = \hat{l}_j + \hat{g} \end{cases} \qquad \begin{cases} \eta_j + i\xi_j = (\hat{\eta}_j + i\hat{\xi}_j)e^{-i\hat{g}} & \\ p_k + iq_k = (\hat{p}_k + i\hat{q}_k)e^{-i\hat{g}} & k \neq N-1 \\ p_{N-1} + iq_{N-1} = Re^{-i\hat{g}} & \end{cases}$$

$$R = \sqrt{2(\sum_{j=1}^{N} \Lambda_j - G_0 - \sum_{j=1}^{N} \frac{\hat{\eta}_j^2 + \hat{\xi}_j^2}{2} - \sum_{j=1}^{N-2} \frac{\hat{p}_j^2 + \hat{q}_j^2}{2})}$$

Similar reductions have been studied in [Malige, Robutel, Laskar 2002], [Herman 2009].

The Fully Reduced System (Proof of Theorem B)

$$H_{fr} = h_{Kep}(\Lambda) + m f_{fr}(\Lambda, \hat{l}, \hat{z})$$

$$\hat{z} = (\hat{\eta}_1, \cdots, \hat{\eta}_n, \hat{p}_1, \cdots, \hat{p}_{N-2}, \hat{\xi}_1, \cdots, \hat{\xi}_N, \hat{q}_1, \cdots, \hat{q}_{N-2})$$

$$(f_{fr})_{av} \sim \hat{C}_0 + \sum_{i=1}^{N} (\hat{s}_i \frac{\eta_i^2 + \xi_i^2}{2} + \sum_{i=1}^{N-2} \hat{s}'_i \frac{p_i^2 + q_i^2}{2} + O(\eta, \xi, p, q)^4 + O(r^2)$$

- The $\hat{s}_i$, $\hat{s}'_j$ are given by $\hat{s}_i = s_i - s_N$, $\hat{s}'_j = s_j - s_N$;
- They may do not verify resonances for well separated semi--major axises;
- The system looses D'Alembert rules and has a singularity for co--circular and co--inclined configuration;
- Using Birkhoff Theory and (Poschel) Normal Form Theory suitably adapted to degenerate system [Biasco, Chierchia, Valdinoci, 2003] the system is conjugated to $H_{NF} = h_{Kep}(\Lambda) + mN(\Lambda, \hat{z}) + mO(\bar{e}_0^{2p+1})$, where $N(\Lambda, \hat{z})$ is a polynomial of degree p in $\frac{\hat{\eta}_1^2 + \hat{\xi}_1^2}{2}$, ...

Sketch of Proof of Theorem C

- First step: reduce the SO(3)-symmetry. The system has three degrees of freedom: two fast and one slow;
- Second step: consider the integrable truncation

$$h_{Trunc} = h_{Kep}(\Lambda) + m(C_0(\Lambda) + \hat{s}J + \frac{1}{2}\hat{T}(\Lambda)J^2 + \frac{1}{6}\hat{S}(\Lambda)J^3)$$

  and prove steepness (Nekhorossev's three--jet condition is satisfied).
- Quantify the steepness parameters in terms of m, $\frac{\overline{a}_1}{\underline{a}_2}$ and e_{min}.

Sketch of Proof of Theorem C

- **Third step**: apply averaging Theory and Birkhoff Theory to reduce the system to

$$h_{Trunc} + mO(e^{-K}) + mO(e_{min}^{2p+1})$$

  where K is the ultraviolet cutoff, to be chosen appropriately.

- **Fourth step**: apply averaging a quantitative version of the Nekhorossev Theorem, in order to quantify parameters. For example, [Nekhorossev, 1979].

On the Proof of Theorem D

- Consider the semi--axes ratio expansion of the averaged perturbation

$$(f_{pr})_{av} = (f_{pr})_{av}^{(0)} + (f_{pr})_{av}^{(2)} + \cdots$$

- the term of order 0 of this expansion depends only on Λ;

- the linear term vanishes.

- the first non trivial term is the averaged quadrupolar potential

$$(f_{pr})_{av}^{(2)} = -\sum_{1\le i<j\le N} \frac{\bar{m}_i \bar{m}_j}{(2pi)^2} \int_{T^2} \frac{\big(3(x^{(j)}\cdot x^{(i)})^2 - |x^{(i)}|^2 |x^{(j)}|^2\big) d\bar{\bar{l}}_i}{|x^{(j)}|^5}$$

where

$$x^{(i)} = x^{(i)}(\Lambda, \bar{\bar{l}}_i, \bar{z})$$

Properties of $(f_{pr})_{av}^{(2)}$

Proposition

- In the case of the planar problem, $(f_{pr})_{av}^{(2)}$ depends only on $\frac{\eta_i^2+\xi_i^2}{2}$ (is integrable and already in Birkhoff Normal form). An explicit expression of it is available.

- $(f_{pr})_{av}^{(2)}$ commutes with

$$\sum_{i=1}^{N-1} \frac{\eta_i^2 + \xi_i^2}{2} + \frac{p_i^2 + q_i^2}{2} \quad \text{and} \quad \frac{\eta_N^2 + \xi_N^2}{2}$$

-> For N=2, spatial $(f_{pr})_{av}^{(2)}$ is integrable [Lidov, Ziglin, 1976]; [Lei Zhao; PhD Thesis, 2013]; used also in [Palacian et al, 2012].

- The proposition follows from the following formula [P. 2013 preprint]

$$(f_{pr})^{(2)}_{av} = -\sum_{1\le i<j\le N} \bar{m}_i\bar{m}_j \frac{M_j m_j^2}{4} \frac{\frac{1}{2pi}\int_T \left(3(C^{(j)}\cdot x^{(i)})^2 - |x^{(i)}|^2|C^{(j)}|^2\right)d\bar{\bar{l}}_i}{|C^{(j)}|^4} \times \left(\frac{1}{2pi}\int_T \frac{d\bar{\bar{l}}_j}{|x^{(j)}|^2}\right)$$

with

$$x^{(i)} = x^{(i)}(\Lambda, \bar{\bar{l}}_i, \bar{z}) \ , \quad C^{(i)} = C^{(i)}(\Lambda, \bar{z}) = x^{(i)}(\Lambda, \bar{\bar{l}}_i, \bar{z}) \times y^{(i)}(\Lambda, \bar{\bar{l}}_i, \bar{z})$$

- In the **planar case** $C^{(j)}\cdot x^{(i)} = 0$

$$(f_{pl})^{(2)}_{av} = \frac{1}{4}\sum_{1\le i<j\le n} \bar{m}_i\bar{m}_j \frac{a_i^2}{a_j^3} \frac{1+\frac{3}{2}e_i^2}{\left(1-\frac{\eta_j^2+\xi_j^2}{2\Lambda_j}\right)^2} \frac{1}{2pi}\int_T \frac{d\zeta}{1-e_j\cos\zeta}$$

with $e_i = \sqrt{\frac{\eta_i^2+\xi_i^2}{\Lambda_i} - (\frac{\eta_i^2+\xi_i^2}{2\Lambda_i})^2}$. (The integral is even in e_j)

A Simplified FT (after V. I. Arnold) and Proof of Theorem D

Lemma[P. 2013] If $H = H_0(I) + mP(I, x, u, v)$, real analytic on $D = V \times T^{n_1} \times B_0^{2n_2}$ satisfies

- H_0 is a diffeomorphism on its domain $V \subset R^{n_1}$;
- $P_{av}(I, u, v) = \bar{P}(I, \frac{u_1^2+v_1^2}{2}, \cdots, \frac{u_{n_2}^2+v_{n_2}^2}{2}) + \tilde{P}(I, u, v)$, $|\tilde{P}(I, u, v)| \leq k$
- $\det D_I^2 H_0(I) \neq 0 \neq \det D_J^2 \bar{P}(I, J)$ on $V \times B^{2n_2}$

for

$$m < m_0 \ , \quad k < k_0 \ , \quad m < \frac{1}{C}(\log(K^{-1}))^{-4b}$$

one can find a Kolmogorov set $K_{m,k} \subset D$ satisfying

$$\text{meas}K_{m,k} \geq (1 - C\sqrt[4]{m}(\log(k^{-1}))^b - C\sqrt{k})\text{meas}D$$

- This lemma refines an analogue lemma in [Arnold, 1963].
- Theorem D follows taking $k = (\frac{\bar{a}_1}{\underline{a}_2})^3$, $H_0 = h_{Kep}$, $\bar{P} = (f_{pr})_{av}^0 + (f_{pr})_{av}^2$.

Future Perspectives

Proving stability for exponentially--long times in the general case is definitely related to prove the following

Conjecture For any N there exists r such that the integrable truncation of order r of the normal form associated to

$$\mathrm{h}_{\mathrm{Kep}}(\Lambda) + \mathrm{m}\Big((\mathrm{f}_{\mathrm{pr}})_{\mathrm{av}}^{(0)} + (\mathrm{f}_{\mathrm{pr}})_{\mathrm{av}}^{(2)}\Big)$$

(eventually reducing the SO(3)--symmetry completely) is steep.

thanks for Your attention