Oligomorphic permutation groups: growth rates and algebras

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The definition

Let $G$ be a permutation group on an infinite set $\Omega$. Then $G$ has a natural induced action on the set of all $n$-tuples of elements of $\Omega$, or on the set of $n$-tuples of distinct elements of $\Omega$, or on the set of $n$-element subsets of $\Omega$. It is easy to see that if there are only finitely many orbits on one of these sets, then the same is true for the others.
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We denote the number of orbits on all $n$-tuples, resp. $n$-tuples of distinct elements, $n$-sets, by $F_n^*(G), F_n(G), f_n(G)$ respectively.
Examples, 1

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- $f_n(A) = 1$;
- $F_n(A) = n!$;
- $F_n^*(A)$ is the number of preorders of an $n$-set.
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Consider $S^r$ acting on $\Omega^r$. Then $F_n^*(S^r) = B(n)^r$. 

For $A_2$ acting on $Q^2$, $f_n(A_2)$ is the number of zero-one matrices (of unspecified size) with $n$ ones and no rows or columns of zeros.
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Consider $S^r$ acting on $\Omega_r$. Then $F^*_n(S^r) = B(n)^r$. From this we can find $F_n(S^r)$ by inversion:

$$F_n(G) = \sum_{k=1}^{n} s(n,k) F^*_k(G)$$

for any oligomorphic group $G$, where $s(n,k)$ is the signed Stirling number of the second kind.
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Let $G = S_2 \text{Wr} A$, where $S_2$ is the symmetric group of degree 2. Then $f_n(G)$ is the $n$th Fibonacci number.
Examples, 4

There is a unique countable random graph $R$: that is, if we choose a countable graph at random (edges independent with probability $\frac{1}{2}$), then with probability 1 it is isomorphic to $R$. 

$G = \text{Aut}(R)$, then $F_n(G)$ and $f_n(G)$ are the numbers of labelled and unlabelled graphs on $n$ vertices.
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What other structures can be specified by countability and first-order axioms? Such structures are called countably categorical.
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In fact, more is true: the types over the theory of $M$ are all realised in $M$, and the sets of $n$-tuples which realise the $n$-types are precisely the orbits of $\text{Aut}(M)$ on $M^n$. 
Several things are known about the behaviour of the sequence \((f_n(G))\):

- It is non-decreasing.
- Either it grows like a polynomial (that is, \(a_n \leq f_n(G) \leq b_nk\) for some \(a, b > 0\) and \(k \in \mathbb{N}\)), or it grows faster than any polynomial.
- If \(G\) is primitive (that is, it preserves no non-trivial equivalence relation on \(\Omega\)), then either \(f_n(G) = 1\) for all \(n\), or \(f_n(G)\) grows at least exponentially.
- If \(G\) is highly homogeneous (that is, if \(f_n(G) = 1\) for all \(n\)), then either there is a linear or circular order on \(\Omega\) preserved or reversed by \(G\), or \(G\) is highly transitive (that is, \(F_n(G) = 1\) for all \(n\)).
- There is no upper bound on the growth rate of \((f_n(G))\).
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Examples suggest that much more is true. For any reasonable growth rate, appropriate limits should exist:

- For polynomial growth of degree $k$, $\lim \left( \frac{f_n(G)}{n^k} \right)$ should exist;
- For fractional exponential growth (like $\exp(n^{c})$), $\lim \left( \frac{\log \log f_n(G)}{\log n} \right)$ should exist;
- For exponential growth, $\lim \left( \frac{\log f_n(G)}{n} \right)$ should exist;
- and so on.

I do not know how to prove any of these things; and I do not know how to formulate a general conjecture.
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A Ramsey-type theorem

**Theorem**
Let $X$ be an infinite set, and suppose that the $n$-element subsets of $\Omega$ are coloured with $r$ different colours (all of which are used). Then there is an ordering $(c_1, \ldots, c_r)$ of the colours, and infinite subsets $Y_1, \ldots, Y_r$ of $X$, such that, for $i = 1, \ldots, r$, the set $Y_i$ contains an $n$-set of colour $c_i$ but none of colour $c_j$ for $j > i$. The existence of $Y_1$ is the classical theorem of Ramsey. There is a finite version of the theorem, and so there are corresponding 'Ramsey numbers'. But very little is known about them!
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Monotonicity

Corollary

*The sequence \( (f_n(G)) \) is non-decreasing.*
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Proof.
Let \(r = f_n(G)\), and colour the \(n\)-subsets with \(r\) colours according to the orbits. Then by the Theorem, there exists an \((n + 1)\)-set containing a set of colour \(c_i\) but none of colour \(c_j\) for \(j > i\). These \((n + 1)\)-sets all lie in different orbits; so \(f_{n+1}(G) \geq r\). \qed
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There is also an algebraic proof of this corollary. We’ll discuss this later.
Let \( \binom{\Omega}{n} \) denote the set of \( n \)-subsets of \( \Omega \), and \( V_n \) the vector space of functions from \( \binom{\Omega}{n} \) to \( \mathbb{C} \).
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We make \(A = \bigoplus_{n \geq 0} V_n\) into an algebra by defining, for \(f \in V_n\), \(g \in V_m\), the product \(fg \in V_{n+m}\) by

\[
(fg)(K) = \sum_{M \in \binom{K}{m}} f(M)g(K \setminus M)
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for \(K \in \binom{\Omega}{m+n}\), and extending linearly.

\(A\) is a commutative and associative graded algebra over \(\mathbb{C}\), sometimes referred to as the reduced incidence algebra of finite subsets of \(\Omega\).
A graded algebra, 1

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Now let \( G \) be a permutation group on \( \Omega \), and let \( V_n^G \) denote the set of fixed points of \( G \) in \( V_n \). Put

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If $G$ is oligomorphic, then the dimension of $V_n^G$ is $f_n(G)$, and so the Hilbert series of the algebra $A[G]$ is the ordinary generating function of the sequence $(f_n(G))$. 

What properties does this algebra have?

Note that it is not usually finitely generated since the growth of $(f_n(G))$ is polynomial only in special cases.
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A non-zero-divisor

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This theorem gives another proof of the monotonicity of $(f_n(G))$. For multiplication by $e$ is a monomorphism from $V_n^G$ to $V_{n+1}^G$, and so $f_{n+1}(G) = \dim v_{n+1}^G \geq \dim V_n^G = f_n(G)$. 
An integral domain

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The converse, a long-standing conjecture, has recently been proved by Maurice Pouzet:

**Theorem**

*If $G$ has no finite orbits on $\Omega$, then $\mathcal{A}[G]$ is an integral domain.*
Consequences

Pouzet’s Theorem has a consequence for the growth rate:

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*If G is oligomorphic, then*

\[ f_{m+n}(G) \geq f_m(G) + f_n(G) - 1. \]
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**Proof.**

Multiplication maps \( V^G_m \otimes V^G_n \) into \( V^G_{m+n} \); by Pouzet’s result, it is injective on the projective Segre variety, and a little dimension theory gets the result.
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It seems very likely that better understanding of the algebra $A[G]$ would have further implications for growth rate.
Brief sketch of the proof

Let $\mathcal{F}$ be a family of subsets of $\Omega$. A subset $T$ is transversal to $\mathcal{F}$ if it intersects each member of $\mathcal{F}$. The transversality of $\mathcal{F}$ is the minimum cardinality of a transversal.
Brief sketch of the proof

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Pouzet shows that, if $f \in V_m$ and $g \in V_n$ satisfy $fg = 0$, then the transversality of $\text{supp}(f) \cup \text{supp}(g)$ is finite, and is bounded by a function of $m$ and $n$. (Here $\text{supp}(f)$ denotes the support of $f$.)
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Here is Pouzet’s theorem again:

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The proof of this makes it clear that it is another kind of ‘Ramsey theorem’. If $\tau(m, n)$ denotes the smallest $t$ such that the transversality is at most $t$, then we have the interesting problem of finding $\tau(m, n)$. Pouzet shows that $\tau(m, n) \geq (m + 1)(n + 1) - 1$. On the other hand, the upper bounds coming from his proof are really astronomical!