Crossings, Colorings, and Cliques

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Joint with Mike Albertson and Jacob Fox.

On Graphs with Crossings, CanaDAM
28 May 2009
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[Diagram of a pentagon with edges highlighted]
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[Diagram of a critical graph with vertices and edges indicating the structure of the graph.]
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Thm. (Brooks’ Theorem) If $G$ is connected and not a complete graph or odd cycle, then $\chi(G) \leq \Delta(G)$.

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Hence $m \geq 3n + 1$ and $cr(G) \geq m - (3n - 6) \geq 7$. $lacksquare$
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Easier proofs of harder results

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Thm. [Kostochka-Stiebitz ’96] If $G$ is $r$-critical and $G \neq K_r$ and $n \neq 2r - 1$, then

$$m \geq \frac{r-1}{2}n + r - 3.$$
Proving Albertson’s Conjecture (for lots more cases)

**Crossing Lemma** [Leighton; Ajtai et. al. ’82] If \( m \geq 4n \), then

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\text{cr}(G) \geq \frac{1}{64} \frac{m^3}{n^2}.
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**Crossing Lemma** [Pach et. al. ’06] If \( m \geq \frac{103}{16} n \), then

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**Thm.** Albertson’s Conjecture is true for \( r \leq 12 \).