On Minimum and Maximum Values of γ-labelings of Graphs

Grady Bullington, Linda Eroh, Steven J. Winters
University of Wisconsin Oshkosh

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1. Definitions
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3. Complete Bipartite
4. Spectra of \(K_{n,n}\)
5. Products of Cycles
A \( \gamma \)-labeling of a graph \( G \) with \( m \) edges is a one-to-one function from \( V(G) \) to \( \{0, 1, \ldots, m\} \).

A \( \gamma \)-labeling exists for a graph if the order is at most one more than the size. Any connected graph has a \( \gamma \)-labeling.

A \( \gamma \)-labeling induces a labeling of the edges. The edge \( uv \) is labeled with \( |\text{label}(u) - \text{label}(v)| \).

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The **value** of a $\gamma$-labeling is the sum of the induced labels of the edges.

\[
\text{val}_{\text{max}}(G) = \text{maximum value over all } \gamma\text{-labelings of } G \\
\text{val}_{\text{min}}(G) = \text{minimum value over all } \gamma\text{-labelings of } G
\]

The **spectrum** of $G$ is the set of all possible values of $\gamma$-labelings of $G$. 
**Definitions**

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The spectrum of $G$ is the set of all possible values of $\gamma$-labelings of $G$. 
I recently learned that this is very similar to the bandwidth of a labeling.

If the vertices of $G$ are labeled with $\{1, 2, \ldots, |V(G)|\}$, the bandwidth is the sum of the induced edge labels.
A $\gamma$-labeling will use labels from $\{0, 1, 2, 3, 4, 5\}$.
Example $\text{Val}_{\text{min}}(G)$

A $\gamma$-labeling will use labels from $\{0, 1, 2, 3, 4, 5\}$. 

\begin{center}
\begin{tikzpicture}
    \node (1) at (1,1) {1};
    \node (2) at (1,-1) {2};
    \node (3) at (-1,-1) {3};
    \node (0) at (-1,1) {0};
    \draw (1) -- (0);
    \draw (0) -- (2);
    \draw (3) -- (2);
    \draw (3) -- (0);
\end{tikzpicture}
\end{center}
**Example** \( Val_{\min}(G) \)

A \( \gamma \)-labeling will use labels from \( \{0, 1, 2, 3, 4, 5\} \).
A $\gamma$-labeling will use labels from $\{0, 1, 2, 3, 4, 5\}$. $val_{min}(G) = 7$. 

\[ 
\begin{array}{ccc}
1 & 1 & 0 \\
2 & 1 & 2 \\
3 & 1 & 2 \\
\end{array} 
\]
A $\gamma$-labeling will use labels from \{0, 1, 2, 3, 4, 5\}. 

\[ \text{Example } Val_{max}(G) \]
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\[ \text{Example } Val_{\text{max}}(G) \]
A $\gamma$-labeling will use labels from \{0, 1, 2, 3, 4, 5\}. 
$\text{val}_{\text{max}}(G) = 17$. 

![Diagram showing a $\gamma$-labeling example with labels 0, 4, 4, 5, 1, 3, 4, 1.]
\[ \text{val}_{\text{min}}(K_{n_1,n_2}) \]

For \( n_1 \geq n_2 \),
\[ \text{val}_{\text{min}}(K_{n_1,n_2}) = \frac{n_2(2n_2^2+1)}{3} + (n_1 - n_2)n_2^2 + \left\lceil \frac{(n_1-n_2)^2}{4} \right\rceil n_2 \]
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$\text{val}_{min}(K_{n_1,n_2})$

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\( \text{val}_{\text{min}}(K_{n_1,n_2,\ldots,n_r}) \)

A similar labeling produces the minimum value for a complete multipartite graph. (Some edges left out for clarity)
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Sketch of proof: WLOG, labels are consecutive and start at 0. Given any labeling with these labels, it can be converted into this labeling by a series of swaps which does not increase the value.
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$\text{val}_{\text{max}}(K_{n_1,n_2}) = n_1 n_2 (n_1 n_2 - \frac{1}{2} n_1 - \frac{1}{2} n_2 + 1)$
For every even positive integer $n$, every number in the spectrum of $K_{n,n}$ is even.

Sketch of proof: First, we show that any $\gamma$-labeling of $K_{n,n}$ can be converted into any other $\gamma$-labeling by a series of moves, where each move is one of the following:

- add or subtract 1 from a single label
- swap two consecutive labels used in opposite partite sets

Then we show that each of these moves changes the value by an even number.
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Then we show that each of these moves changes the value by an even number.
For every even positive integer $n$, every even integer from $\frac{n(2n^2+1)}{3}$ to $\frac{n^2(2n^2-n+2)}{4}$ is in the spectrum of $K_{n,n}$.

From previous results, $\text{val}_{\text{min}}(K_{n,n}) = \frac{n(2n^2+1)}{3}$ and $\text{val}_{\text{max}}(K_{n,n}) = n^2(n^2 - n + 1)$. 
For any odd integer $n \geq 3$, every integer from $\frac{n(n+1)(3n-1)}{4}$ to $\frac{n(2n^3+n^2-2n+3)}{4}$ is in the spectrum of $K_{n,n}$.

From previous results, $\text{val}_{\min}(K_{n,n}) = \frac{n(2n^2+1)}{3}$ and $\text{val}_{\max}(K_{n,n}) = n^2(n^2 - n + 1)$. 
For any integer $n \geq 2$, the value $\frac{n(2n^2+1)}{3} + 1$ is not in the spectrum of $K_{n,n}$. Furthermore, the values $n^2(n^2 - n + 1) - i$, for $1 \leq i \leq n - 1$, are not in the spectrum of $K_{n,n}$.
If $a$ and $b$ are even integers, $a \geq 4$ and $b \geq 4$, then
\[ \text{val}_{\text{max}}(C_a \times C_b) = ab(3ab + 2). \]
\( \text{Val}_{\text{max}}(C_a \times C_b) \)

If \( a \geq 3 \) is odd and \( b \geq 4 \) is even, then

\[
\text{val}_{\text{max}}(C_a \times C_b) = 3a^2b^2 - ab^2 + 2ab - \frac{1}{2}b^2 - b.
\]
If $a \geq 3$ and $b \geq 3$ are odd integers, then

$$\text{val}_{\text{max}}(C_a \times C_b) = 3a^2b^2 - ab^2 - a^2b + 2ab - a - b - \frac{1}{2}(a^2 + b^2).$$
If \( a \geq b \geq 3 \) are integers, then
\[
\text{val}_{\text{min}}(C_a \times C_b) = 2a(b - 1) + 2b^2(a - 1).
\]
A SAMPLING OF OTHER RESULTS ON \(\gamma\)-LABELING

G. Chartrand, D. Erwin, D.W. VanderJagt, and P. Zhang found the spectrum of stars \(K_{1,n}\) and the maximum and minimum value of paths, cycles, and complete graphs.

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A sampling of other results on $\gamma-$labeling

Futaba Okamoto, Ping Zhang, and Varaporn Saenpholphat defined balanced $\gamma-$labeling of digraphs and characterized which digraphs have balanced $\gamma-$labelings.

Grady Bullington showed that the Connell sum sequence is a sharp upper bound on the maximum value of a $\gamma-$labeling of a graph of given order.

S.M. Hedge and P. Shankaran defined and studied analogous results for edge sum labeling.
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