# Variations on Pebbling and Graham's Conjecture

David S. Herscovici
Quinnipiac University
with Glenn H. Hurlbert
and Ben D. Hester
Arizona State University

# Talk structure

- Pebbling numbers
- Products and Graham's Conjecture
- Variations
  - Optimal pebbling
  - Weighted graphs
  - Choosing target distributions

# Basic notions

Distributions on G:
D:V(G)→ℕ

D(v) counts pebbles on v

Pebbling moves

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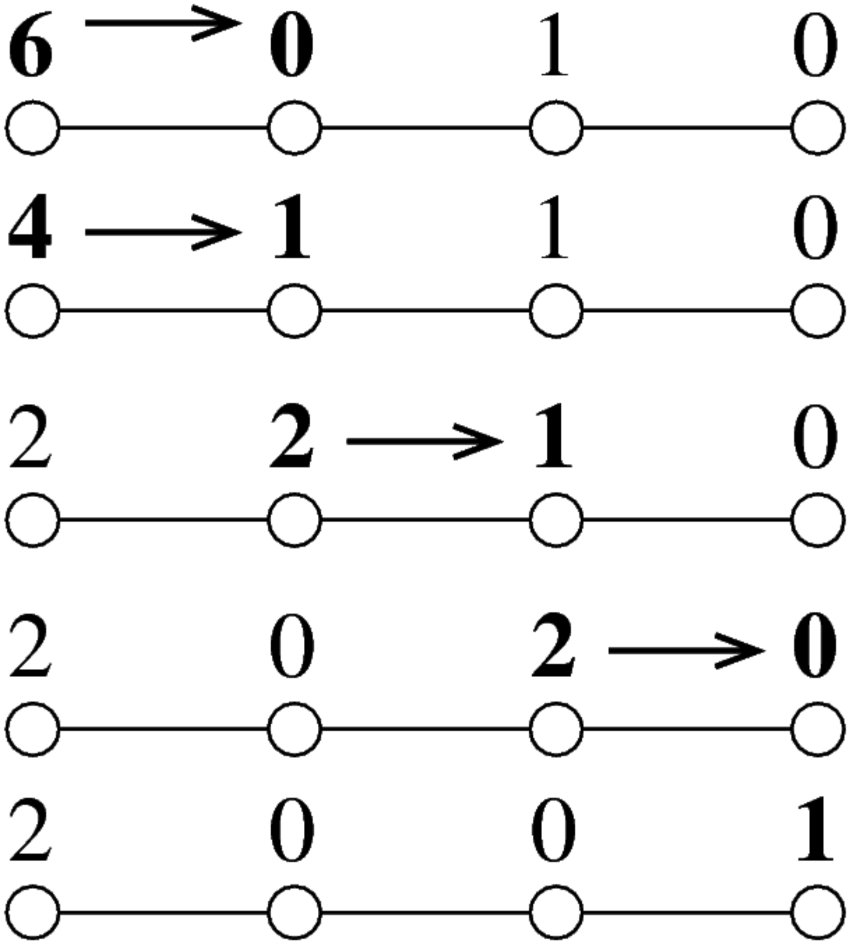
Pebbling moves



# Pebbling numbers

$\pi(G, D)$ is the number of pebbles required to ensure that D can be reached from any distribution of $\pi(G, D)$ pebbles.

If S is a ***set*** of distributions on G

$$\pi(G, S) = max_{D \in S} \pi(G, D)$$

# Pebbling numbers

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If S is a ***set*** of distributions on G

$$\pi(G,S)=max_{D\in S}\pi(G,D)$$

Optimal pebbling number:

π*(G, S) is the number of pebbles required in ***some*** distribution from which every D∈S can be reached

# Common pebbling numbers

$S_1(G)$: 1 pebble anywhere

$$\pi(G)=\pi(G,S_1(G))$$

(*pebbling number*)

$S_t(G)$: t pebbles on some vertex

$$\pi_t(G)=\pi(G,S_t(G))$$

(*t-pebbling number*)

$\delta_v$: One pebble on v

$$\pi(G,v)=\pi(G,\delta_v)$$

# S(G, t) and π(G, t)

S(G, t): all distributions with a total of t pebbles (anywhere on the graph)

**Conjecture**

$$\pi(G, S(G, t)) = \pi(G, S_t(G))$$

i.e. hardest-to-reach ***target*** configurations have all pebbles on one vertex

True for $K_n$, $C_n$, trees

# Cover pebbling

Γ(G): one pebble on every vertex

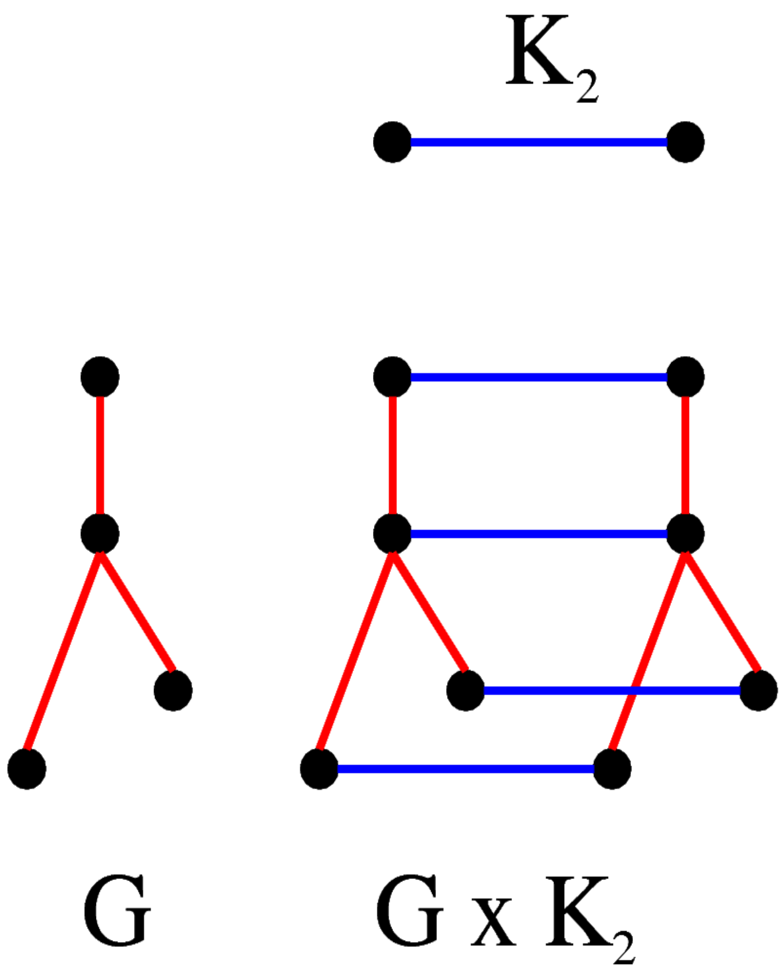
$$\gamma(G) = \pi(G, \Gamma(G))$$

(*cover pebbling number*)

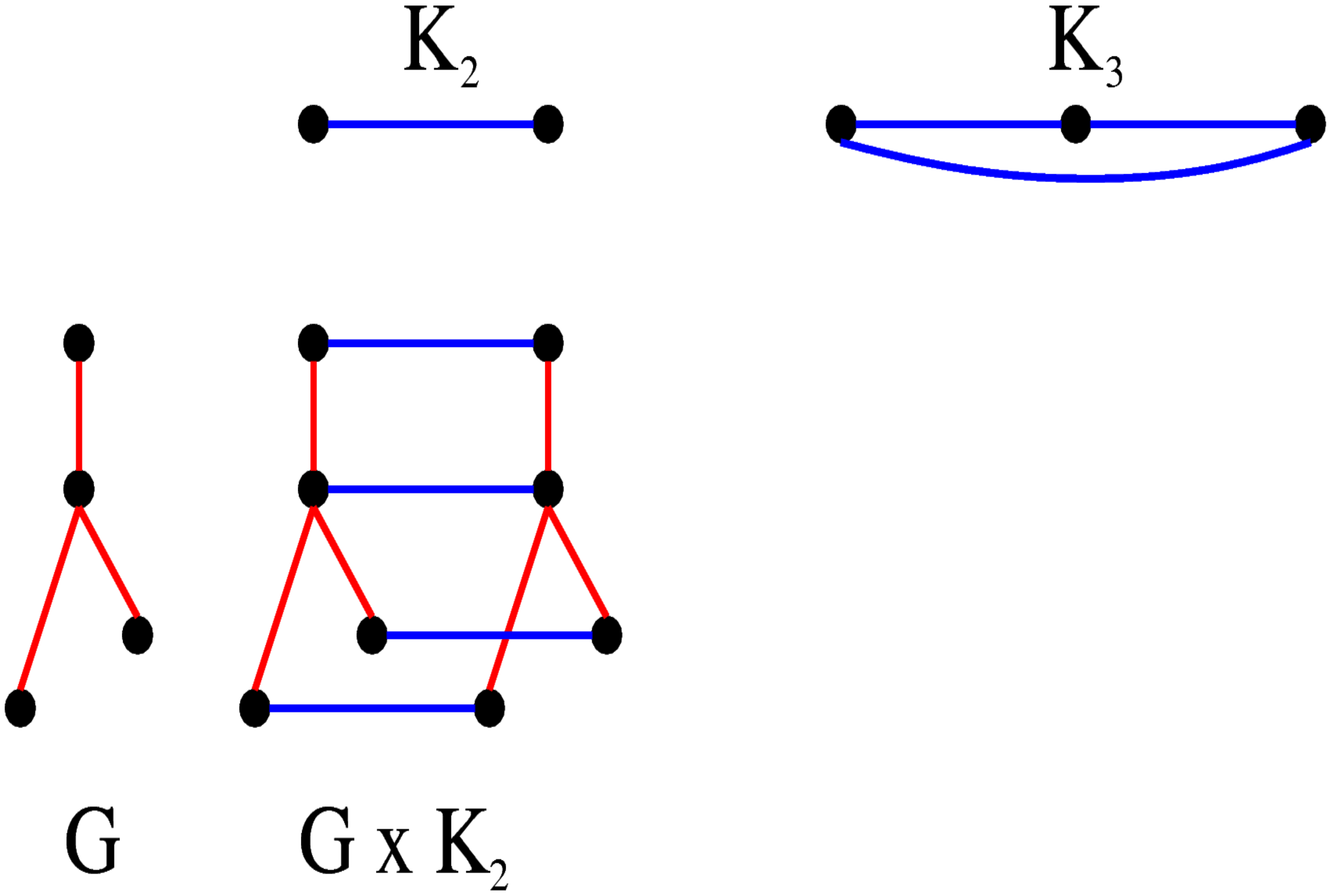
Sjöstrund: If D(v) ≥ 1 for all vertices v, then there is a critical distribution with all pebbles on one vertex.

(A *critical* distribution has one pebble less than the required number and cannot reach some target distribution)

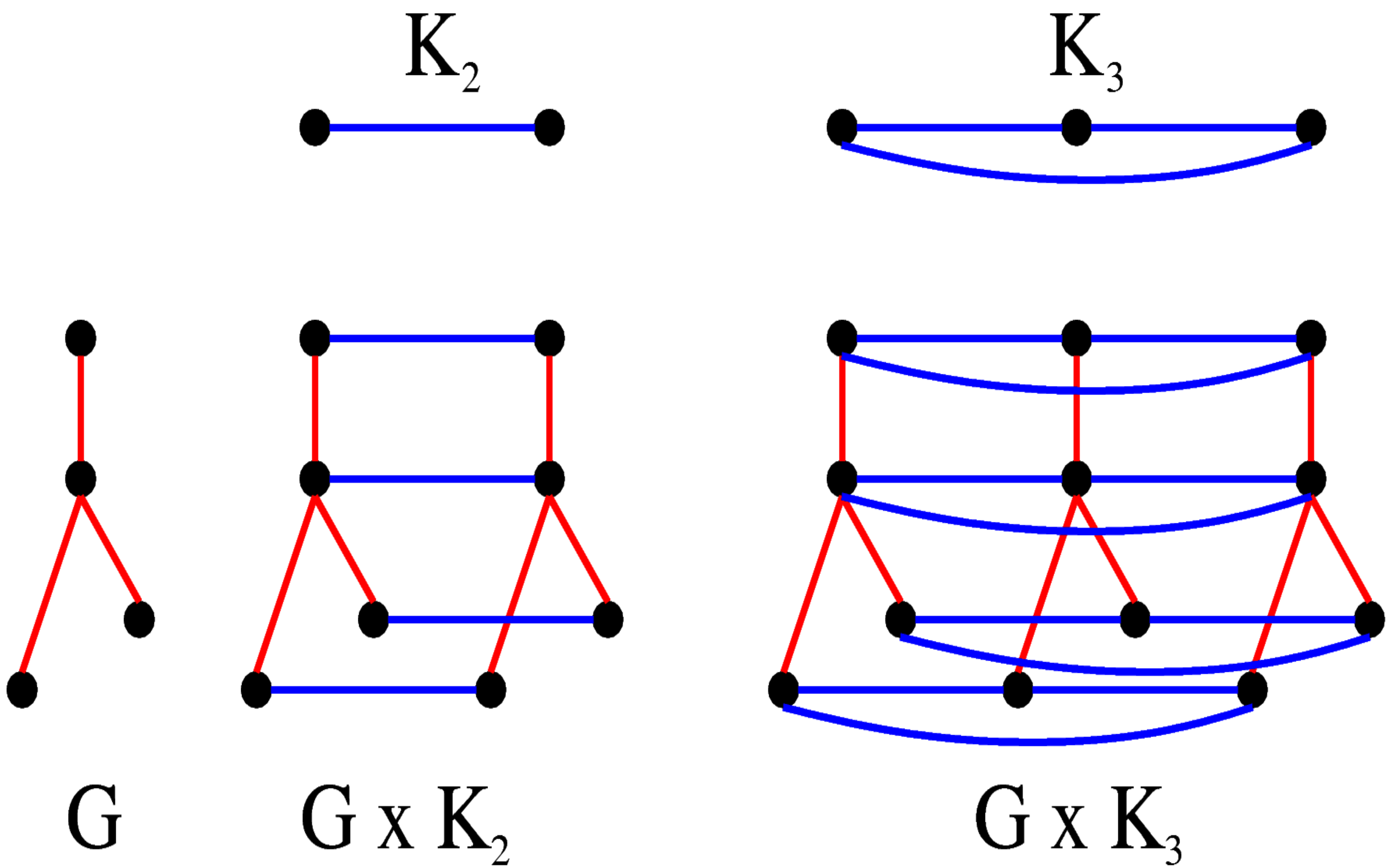
# Cartesian products of graphs



# Cartesian products of graphs



# Cartesian products of graphs



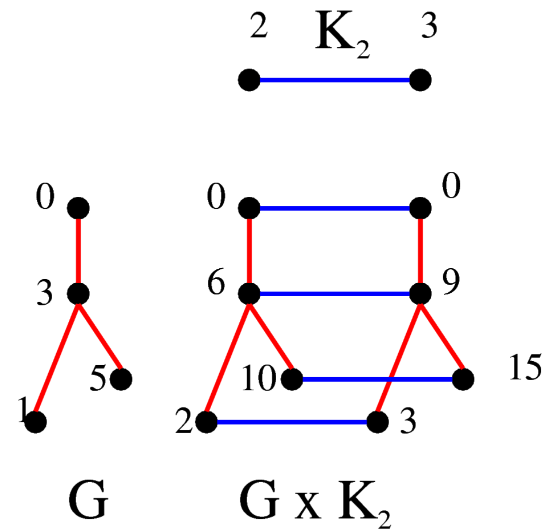
# Products of distributions

Product of distributions:

D1 on G; D2 on H

then D1·D2 on GxH

$$D_1 \cdot D_2((v,w)) = D_1(v)\, D_2(w)$$



# Products of distributions

Product of distributions:

$D_1$ on G; $D_2$ on H

then $D_1 \cdot D_2$ on GxH

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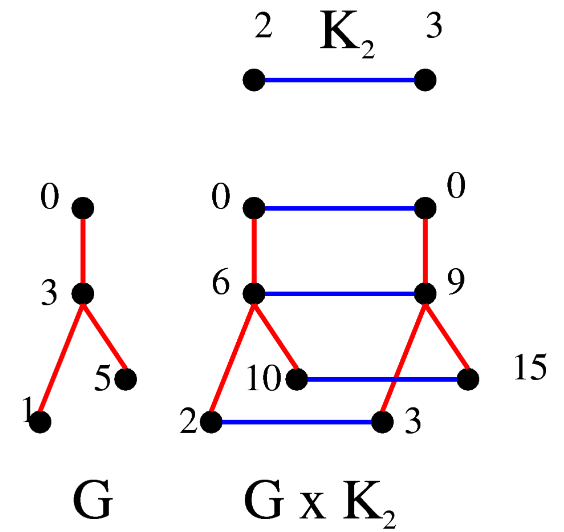
Products of ***sets*** of distributions

$S_1$ a set of distros on G

$S_2$ a set of distros on H

$$S_1 \cdot S_2 = \{D_1 \cdot D_2 : D_1 \in S_1 ; D_2 \in S_2\}$$



# Graham's Conjecture generalized

Graham's Conjecture:

$$\pi(G \times H) \leq \pi(G)\pi(H)$$

Generalization:

$$\pi(G \times H, S_1 \cdot S_2) \leq \pi(G, S_1)\pi(H, S_2)$$

# Optimal pebbling

**Observation (not obvious):**

If we can get from $D_1$ to $D_1'$ in G and from $D_2$ to $D_2'$ in H, we can get from $D_1 \cdot D_2$ to $D_1 \cdot D_2'$ in GxH to $D_1' \cdot D_2'$ in GxH

**Conclusion (optimal pebbling):**

$$\pi^*(G \times H, S_1 \cdot S_2) \leq \pi^*(G, S_1)\pi^*(H, S_2)$$

Graham's Conjecture holds for optimal pebbling in most general setting

# Weighted graphs

Edges have weights w

Pebbling moves: remove w pebbles from one vertex, move 1 to adjacent vertex

π(G), π(G, D) and π(G, S) still make sense

GxH also makes sense:

wt({(v,w), (v,w')}) = wt({w, w'})

wt({(v,w), (v',w)}) = wt({v, v'})

# The Good news

Chung: Hypercubes ($K_2$ x $K_2$ x … x $K_2$) satisfies Graham's Conjecture for any collection of weights on the edges

We can focus on complete graphs in most applications

# The Bad News

Complete graphs are hard!

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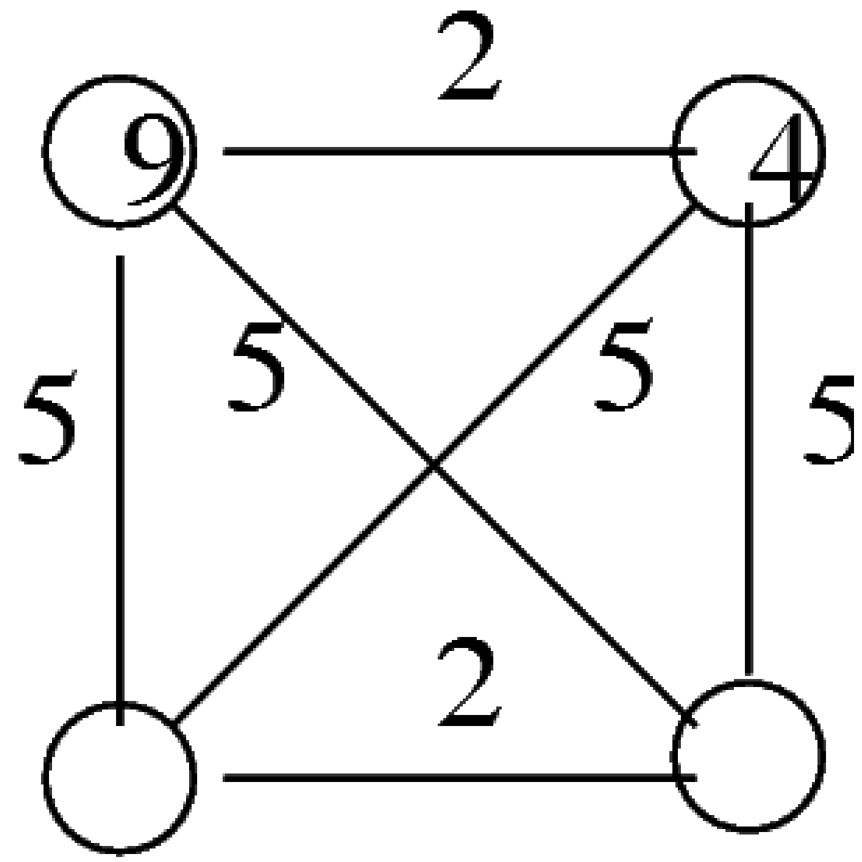
Complete graphs are hard!

Sjöstrund's Theorem fails

13 pebbles on one vertex can cover $K_4$, but...



# Some specializations

Conjecture 1:

$$\pi(G\times H,(v,w))\leq\pi(G,v)\pi(H,w)$$

Conjecture 2:

$$\pi_{st}(G\times H,(v,w))\leq\pi_s(G,v)\pi_t(H,w)$$

Clearly Conjecture 2 implies Conjecture 1

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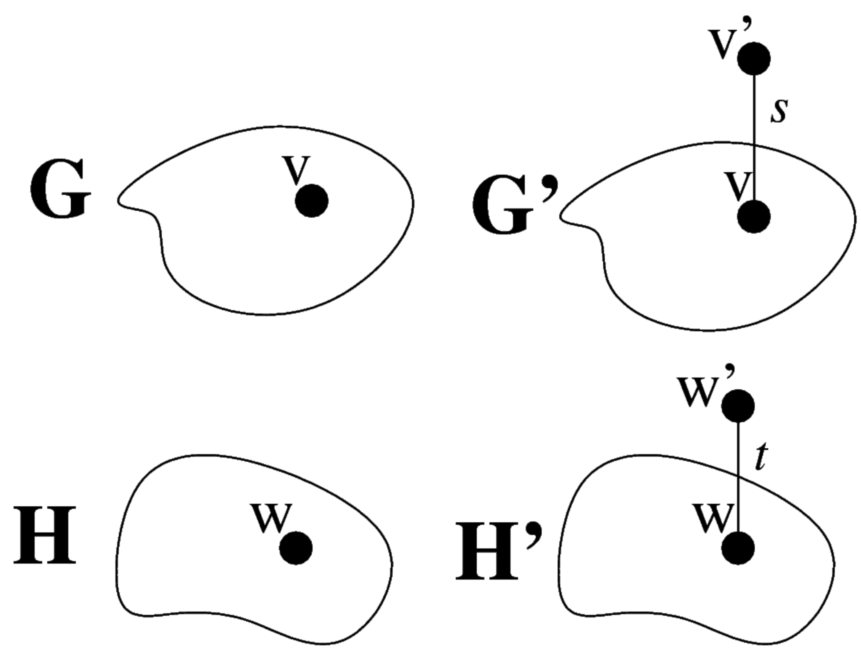
Clearly Conjecture 2 implies Conjecture 1

**These conjectures are equivalent on weighted graphs**

$$\pi(G \times H, (v,w)) \le \pi(G, v)\, \pi(H, w)$$

implies

$$\pi_{st}(G \times H, (v,w)) \le \pi_s(G, v)\, \pi_t(H, w)$$




$$\pi(G', v') = \pi_S(G, v)$$

$$\pi(H', w') = \pi_S(H, w)$$

## $\pi(G\text{x}H, (v,w)) \leq \pi(G, v)\, \pi(H, w)$
## implies
## $\pi_{st}(G\text{x}H, (v,w)) \leq \pi_s(G, v)\, \pi_t(H, w)$

If st pebbles cannot be moved to (v, w) from D in GxH, then (v', w') cannot be reached from D in G'xH' (delay moves onto {v'}xH' and G'x{w'} as long a possible)

$$\pi_{st}(G\times H,(v,w))\leq\pi(G'\times H',(v',w'))$$

$$\leq\pi(G',v')\pi(H',w')=\pi_s(G,v)\pi_t(H,w)$$

# Implications for regular pebbling

Conjecture 1:

$$\pi(G\times H,(v,w))\leq\pi(G,v)\pi(H,w)$$

equivalent to Conjecture 1':

$$\pi_{2^{ab}}(G\times H,(v,w))\leq\pi_{2^a}(G,v)\pi_{2^b}(H,w)$$

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equivalent to Conjecture 1':

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Conjecture 2:

$$\pi_{st}(G \times H,(v,w)) \leq \pi_s(G,v)\pi_t(H,w)$$

equivalent to Conjecture 2':

$$\pi_{st}(G \times H,(v,w)) \leq \pi_s(G,v)\pi_t(H,w)$$

**when s and t are odd**

# Choosing a target

**Observation:** To reach an unoccupied vertex v in G, we need to put two pebbles on any neighbor of v.

We can choose the target neighbor

If S is a *set* of distributions on G, ρ(G, S) is the number of pebbles needed to reach ***some*** distribution in S

**Idea:** Develop an induction argument to prove Graham's conjecture

# Comparing pebbling numbers

$$\forall D \in S \, \forall D' \in S(G, \pi(G, S))$$

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$$\forall D \in S \, \exists D' \in S(G, \pi^*(G,S))$$

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$\forall D \in S \, \forall D' \in S(G, \pi(G, S))$
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$\forall D \in S \, \exists D' \in S(G, \pi^*(G, S))$
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$\forall D' \in S(G, \rho(G, S)) \, \exists D \in S$
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# Properties of ρ(G, S)

$$\rho_1(G) = 1$$

$$\rho_2(G) = |V(G)| + 1$$

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**SURPRISE!**

**Graham's Conjecture fails!**

Let H be the trivial graph: $S_H = \{2\delta_v\}$

$$\rho(G \times H, S_1(G) \cdot S_H) = \rho_2(G) = |V(G)| + 1$$
$$\rho(G, S_1(G)) = \rho_1(G) = 1$$
$$\rho(H, S_H) = 2$$