Variations on Pebbling and Graham's Conjecture

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Talk structure

Pebbling numbers
Products and Graham's Conjecture
Variations
  Optimal pebbling
  Weighted graphs
Choosing target distributions
Basic notions

Distributions on $G$:
$$D: V(G) \rightarrow \mathbb{N}$$

$D(v)$ counts pebbles on $v$

Pebbling moves

6 0 1 0
Basic notions

Distributions on G: $D: V(G) \rightarrow \mathbb{N}$

$D(v)$ counts pebbles on $v$

Pebbling moves

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Pebbling numbers

\( \pi(G, D) \) is the number of pebbles required to ensure that \( D \) can be reached from any distribution of \( \pi(G, D) \) pebbles.

If \( S \) is a set of distributions on \( G \)

\[ \pi(G, S) = \max_{D \in S} \pi(G, D) \]
Pebbling numbers

$\pi(G, D)$ is the number of pebbles required to ensure that $D$ can be reached from any distribution of $\pi(G, D)$ pebbles.

If $S$ is a set of distributions on $G$

$\pi(G, S) = \max_{D \in S} \pi(G, D)$

Optimal pebbling number:
$\pi^*(G, S)$ is the number of pebbles required in some distribution from which every $D \in S$ can be reached.
Common pebbling numbers

$S_1(G)$: 1 pebble anywhere

\[ \pi(G) = \pi(G, S_1(G)) \]

(*pebbling number*)

$S_t(G)$: $t$ pebbles on some vertex

\[ \pi_t(G) = \pi(G, S_t(G)) \]

(*$t$-pebbling number*)

$\delta_v$: One pebble on $v$

\[ \pi(G, v) = \pi(G, \delta_v) \]
$S(G, t)$ and $\pi(G, t)$

$S(G, t)$: all distributions with a total of $t$ pebbles (anywhere on the graph)

Conjecture

\[ \pi(G, S(G, t)) = \pi(G, S_t(G)) \]

i.e. hardest-to-reach target configurations have all pebbles on one vertex

True for $K_n$, $C_n$, trees
Cover pebbling

Γ(G): one pebble on every vertex

\( \gamma(G) = \pi(G, \Gamma(G)) \)

(cover pebbling number)

Sjöstrund: If \( D(v) \geq 1 \) for all vertices \( v \), then there is a critical distribution with all pebbles on one vertex.

(A critical distribution has one pebble less than the required number and cannot reach some target distribution)
Cartesian products of graphs

$K_2$

$G$

$G \times K_2$
Cartesian products of graphs

\[ K_2 \]

\[ K_3 \]

\[ G \]

\[ G \times K_2 \]
Cartesian products of graphs

\( K_2 \)

\( K_3 \)

\( G \)

\( G \times K_2 \)

\( G \times K_3 \)
Product of distributions:

$D_1$ on $G$; $D_2$ on $H$

then $D_1 \cdot D_2$ on $G \times H$

$$D_1 \cdot D_2((v, w)) = D_1(v) D_2(w)$$
Products of distributions

Product of distributions:
D₁ on G; D₂ on H
then D₁ · D₂ on G x H
\[ D_1 \cdot D_2((v, w)) = D_1(v) D_2(w) \]

Products of sets of distributions
S₁ a set of distros on G
S₂ a set of distros on H
\[ S_1 \cdot S_2 = \left\{ D_1 \cdot D_2 : D_1 \in S_1 ; D_2 \in S_2 \right\} \]
Graham's Conjecture generalized

Graham's Conjecture:
\[ \pi(G \times H) \leq \pi(G) \pi(H) \]

Generalization:
\[ \pi(G \times H, S_1 \cdot S_2) \leq \pi(G, S_1) \pi(H, S_2) \]
Optimal pebbling

Observation (not obvious):
If we can get from $D_1$ to $D_1'$ in $G$ and from $D_2$ to $D_2'$ in $H$, we can get from $D_1 \cdot D_2$ to $D_1 \cdot D_2'$ in $G \times H$ to $D_1' \cdot D_2'$ in $G \times H$

Conclusion (optimal pebbling):
$$\pi^*(G \times H, S_1 \cdot S_2) \leq \pi^*(G, S_1) \pi^*(H, S_2)$$

Graham's Conjecture holds for optimal pebbling in most general setting
Weighted graphs

Edges have weights $w$

Pebbling moves: remove $w$ pebbles from one vertex, move 1 to adjacent vertex

$\pi(G)$, $\pi(G, D)$ and $\pi(G, S)$ still make sense

$G \times H$ also makes sense:

$$\text{wt}((\{v, w\}, (v, w')) = \text{wt}(\{w, w'\})$$
$$\text{wt}((\{v, w\}, (v', w)) = \text{wt}(\{v, v'\})$$
The Good news

Chung: Hypercubes \((K_2 \times K_2 \times \ldots \times K_2)\) satisfies Graham's Conjecture for any collection of weights on the edges.

We can focus on complete graphs in most applications.
The Bad News

Complete graphs are hard!
The Bad News

Complete graphs are hard!

Sjöstrund's Theorem fails

13 pebbles on one vertex can cover $K_4$, but...
Some specializations

Conjecture 1:
\[ \pi(G \times H, (v, w)) \leq \pi(G, v) \pi(H, w) \]

Conjecture 2:
\[ \pi_{st}(G \times H, (v, w)) \leq \pi_s(G, v) \pi_t(H, w) \]

Clearly Conjecture 2 implies Conjecture 1
Some specializations

Conjecture 1:
\[ \pi(G \times H, (v, w)) \leq \pi(G, v) \pi(H, w) \]

Conjecture 2:
\[ \pi_{st}(G \times H, (v, w)) \leq \pi_s(G, v) \pi_t(H, w) \]

Clearly Conjecture 2 implies Conjecture 1

These conjectures are equivalent on weighted graphs
\[ \pi(G \times H, (v,w)) \leq \pi(G, v) \pi(H, w) \]
implies
\[ \pi_{st}(G \times H, (v,w)) \leq \pi_s(G, v) \pi_t(H, w) \]

\[ \pi(G', v') = \pi_s(G, v) \]

\[ \pi(H', w') = \pi_s(H, w) \]
\[ \pi(G \times H, (v,w)) \leq \pi(G, v) \pi(H, w) \]

implies

\[ \Pi_{st}(G \times H, (v,w)) \leq \Pi_s(G, v) \Pi_t(H, w) \]

If st pebbles cannot be moved to \((v, w)\) from \(D\) in \(G \times H\), then \((v', w')\) cannot be reached from \(D\) in \(G' \times H'\) (delay moves onto \(\{v'\} \times H'\) and \(G' \times \{w'\}\) as long as possible)

\[
\pi_{st}(G \times H, (v, w)) \leq \pi(G' \times H', (v', w')) \leq \pi(G', v') \pi(H', w') = \pi_s(G, v) \pi_t(H, w)
\]
Implications for regular pebbling

Conjecture 1:
\[ \pi(G \times H, (v, w)) \leq \pi(G, v) \pi(H, w) \]
equivalent to Conjecture 1' :
\[ \pi_{2ab}(G \times H, (v, w)) \leq \pi_{2a}(G, v) \pi_{2b}(H, w) \]
Implications for regular pebbling

Conjecture 1:
\[ \pi(G \times H, (v, w)) \leq \pi(G, v) \pi(H, w) \]
equivalent to Conjecture 1':
\[ \pi_{2ab}(G \times H, (v, w)) \leq \pi_{2a}(G, v) \pi_{2b}(H, w) \]

Conjecture 2:
\[ \pi_{st}(G \times H, (v, w)) \leq \pi_s(G, v) \pi_t(H, w) \]
equivalent to Conjecture 2':
\[ \pi_{st}(G \times H, (v, w)) \leq \pi_s(G, v) \pi_t(H, w) \]
when s and t are odd
Choosing a target

**Observation:** To reach an unoccupied vertex \( v \) in \( G \), we need to put two pebbles on any neighbor of \( v \).

We can choose the target neighbor

If \( S \) is a set of distributions on \( G \), \( \rho(G, S) \) is the number of pebbles needed to reach some distribution in \( S \)

**Idea:** Develop an induction argument to prove Graham's conjecture
Comparing pebbling numbers

$$\forall D \in S \forall D' \in S(G, \pi(G, S))$$

D is reachable from D' by a sequence of pebbling moves
Comparing pebbling numbers

\[ \forall D \in S \forall D' \in S(G, \pi(G, S)) \]
D is reachable from D' by a sequence of pebbling moves

\[ \forall D \in S \exists D' \in S(G, \pi^*(G, S)) \]
D is reachable from D' by a sequence of pebbling moves
Comparing pebbling numbers

∀ \( D \in S \) ∀ \( D' \in S(G, \pi(G, S)) \)
D is reachable from D' by a sequence of pebbling moves

∀ \( D \in S \) ∃ \( D' \in S(G, \pi^*(G, S)) \)
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∀ \( D' \in S(G, \rho(G, S)) \) ∃ \( D \in S \)
D is reachable from D' by a sequence of pebbling moves
Properties of $\rho(G, S)$

\[ \rho_1(G) = 1 \]
\[ \rho_2(G) = |V(G)| + 1 \]
Properties of $\rho(G, S)$

$$\rho_1(G) = 1$$
$$\rho_2(G) = |V(G)| + 1$$

SURPRISE!
Graham's Conjecture fails!
Let $H$ be the trivial graph: $S_H = \{2\delta_v\}$

$$\rho(G \times H, S_1(G) \cdot S_H) = \rho_2(G) = |V(G)| + 1$$
$$\rho(G, S_1(G)) = \rho_1(G) = 1$$
$$\rho(H, S_H) = 2$$