

Star Avoiding Ramsey Numbers

Jonelle Hook, Garth Isaak, Colton Magnant

Department of Mathematics
Lehigh University

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Graph Ramsey Numbers

Example

$$R(C_5, K_4) = 13$$

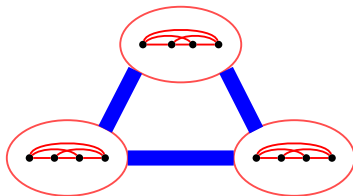
- There exists a 2-coloring of K_{12} with no red C_5 and no blue K_4 .
- Every 2-coloring of K_{13} has a red C_5 or a blue K_4 .

Graph Ramsey Numbers

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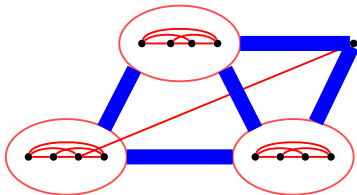
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Example

'Proof' that $R(C_5, K_4) = 13$

- 2 red edges to one part \Rightarrow red C_5
- blue edge to each part \Rightarrow blue K_4

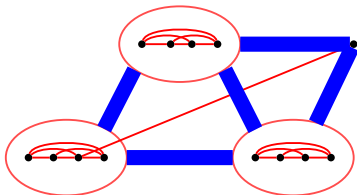


Can color 9 edges but 10th forces red C_5 or K_4

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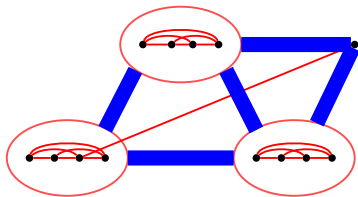


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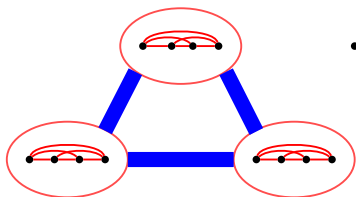
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There are 6 critical colorings (later)

Questions

- When can we classify all sharpness examples for $R(G, H) = r$?
 - What are all good colorings of K_{r-1} (critical colorings)



Questions

- When can we classify all sharpness examples for $R(G, H) = r$?
 - What are all good colorings of K_{r-1} (critical colorings)
- How many edges to the r^{th} vertex must be colored before a red G or blue H is forced?

A second look at our problem:

- Graph Ramsey: smallest r with no good coloring

... K_{r-1} , K_r , K_{r+1} , ...

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- Upper and lower Ramsey for $R(G, H) = r$:

Lower: smallest s with no good coloring for *some* F

Upper: smallest s with no good coloring for *every* F

... $|E(F)| = s - 1$, $|E(F)| = s$, $|E(F)| = s + 1$, ...

Restrict to $|V(F)| = r$

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- Star avoiding Ramsey for $R(G, H) = r$:

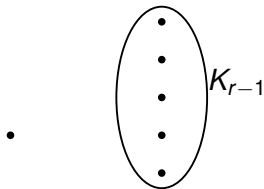
smallest $r - 1 - t$ with no good coloring

... $K_{r-1} \setminus S(1, t-1)$, $K_{r-1} \setminus S(1, t)$, $K_{r-1} \setminus S(1, t+1)$, ...

Star avoiding Ramsey:

$R(G, H) = r$ add/color edges to K_{r-1} one at a time:

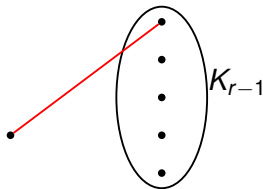
When is a red G or blue H forced?



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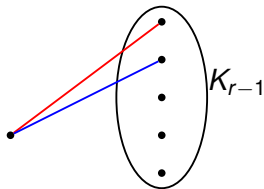
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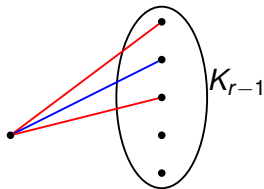
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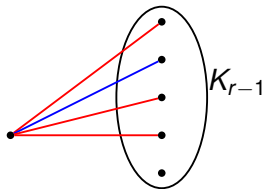
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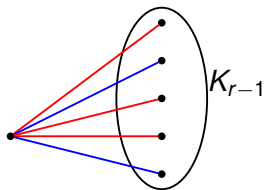
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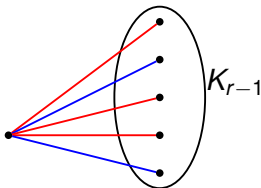
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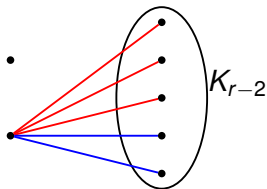
When is a red G or blue H forced?



- Proofs: First classify sharpness examples
Good colorings of K_{r-1}
- Examples with 'few' extra edges needed and with 'many' extra edges needed

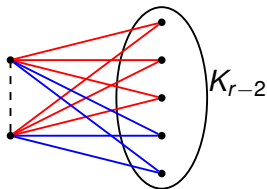
Example

- $R(K_m, K_n) = r$: must add *all* $r - 1$ edges (Chvatal 1974) even though we do not know what r is
-
-



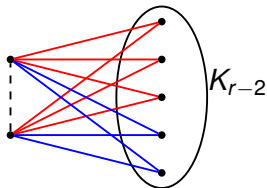
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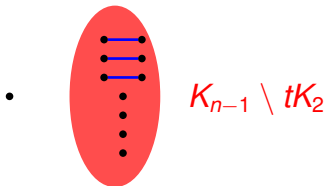
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- $R(K_m, K_n) = r$: must add *all* $r - 1$ edges (Chvatal 1974) even though we do not know what r is
- make a copy of a vertex
- similar for $R(mK_3, mK_3) = 5m$



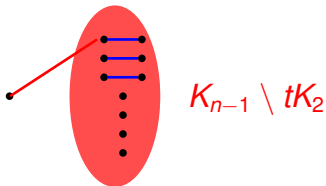
Example ($R(P_n, P_3) = n$ (Gerencser and Gyrafas 1967))

- $R(P_n, P_3) = n$
- Can only add *one* edge to K_{n-1} before a red P_n or blue P_3 is forced.
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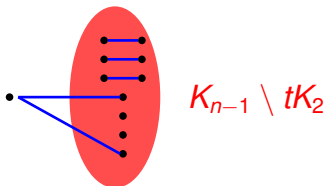
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- Red edge \Rightarrow red P_n
-

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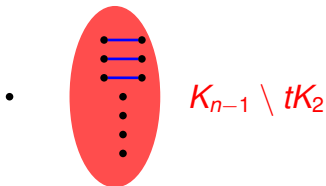
- $R(P_n, P_3) = n$
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-
- Two Blue edges \Rightarrow blue P_3

Example ($R(P_n, P_3) = n$ (Gerencsér and Gyrafas 1967))

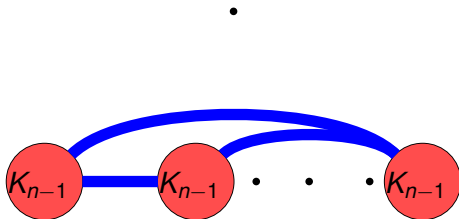
- $R(P_n, P_3) = n$
- Can only add *one* edge to K_{n-1} before a red P_n or blue P_3 is forced.
- Sharpness examples: Blue graph is a matching plus isolated vertices



Example $(R(T_n, K_m) = (n - 1)(m - 2) + 1$ (Chvatal 1977))

- Unique sharpness example:

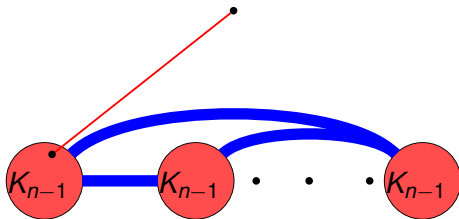
Red graph is $(m - 1)K_{n-1}$ Blue graph is $K_{n-1, n-1, \dots, n-1}$



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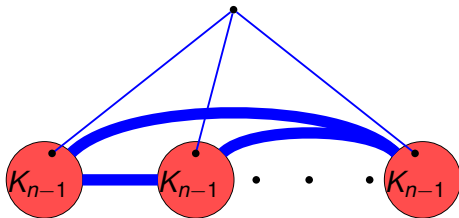
- Red edge \Rightarrow red T_n



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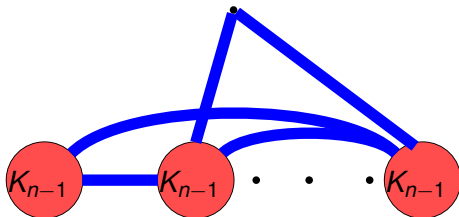
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- Blue edges to all parts \Rightarrow blue K_m

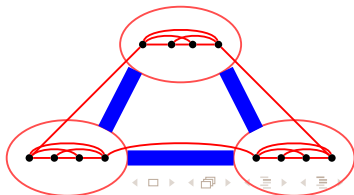
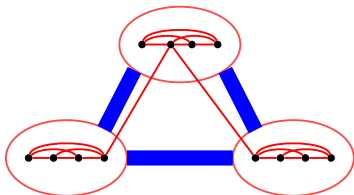
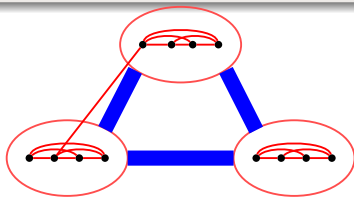
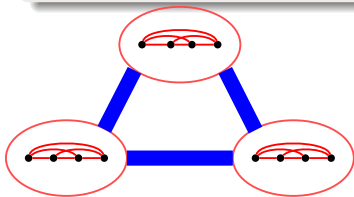
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- Unique sharpness example:
 Red graph is $(m-1)K_{n-1}$ Blue graph is $K_{n-1, n-1, \dots, n-1}$
- (only) add all $(n-1)(m-2)$ blue edges avoiding one part



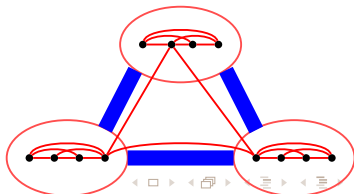
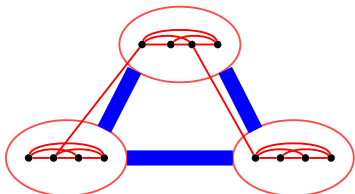
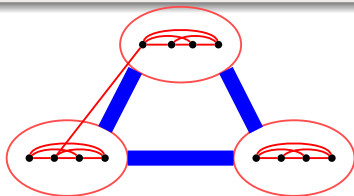
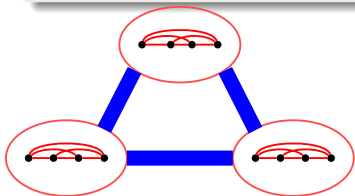
Example ($R(C_5, K_4) = 13$)

- Exactly 6 good colorings of K_{12} (Jayawardene and Rousseau 2000)
- Ends must be different (or same) for 3 extra red edges
- Extends to $R(C_n, K_4) = 3n - 2$ (but not $n = 4$)



Example ($R(C_5, K_4) = 13$)

- Exactly 6 good colorings of K_{12} (Jayawardene and Rousseau 2000)
- Ends must be the same for 3 extra red edges for $n \geq 6$
- Extends to $R(C_n, K_4) = 3n - 2$



Summary of Results

Ramsey number	Minimum Number of edges to force bad coloring
$R(mK_2, mK_2) = 3m - 1$ [L 1984]	m
$R(mK_3, mK_3) = 5m$ [BES 1975]	$5m$
$R(T_n, K_m) = (n - 1)(m - 1) + 1$ [C 1977]	$(n - 1)(m - 2) + 1$
$R(C_n, K_3) = 2n - 1$ [FS 1974]	$n + 1$
$R(C_n, K_4) = 3n - 2$ [SRM 1999]	$2n$
$R(P_n, P_3) = n$ [GG 1967]	2
$R(P_n, P_4) = n + 1$ [GG 1967]	2
$R(P_n, P_5) = n + 1$ [GG 1967]	3
$R(P_n, P_m) = n + \lfloor \frac{m}{2} \rfloor - 1$ [GG 1967] for all $n \geq m \geq 2$	$\lfloor \frac{m}{2} \rfloor$ (probably)