Fractional and integer colourings in claw-free graphs

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supported by AEOLUS project and NSERC

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CanaDAM, May 2009

Overview

Our Motivation

We know $\chi \leq (1 + o(1))\chi_f$ for line graphs.
Can we extend this to broader classes of graphs?

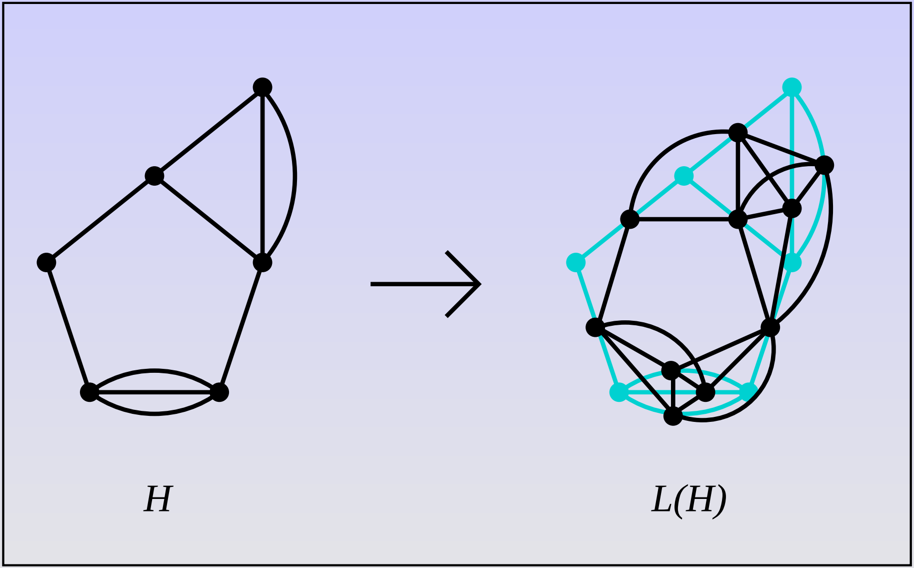
Main Tool: Structure Theorem

$\alpha \geq 4 \ \rightarrow$ graph built as a generalization of a line graph.
We use restricted colourings of the "edges".

New Results

Quasi-line: $\chi \leq \chi_f + 3\sqrt{\chi_f}$.
Claw-free: $\chi \leq \chi_f + c\sqrt{\chi_f}$ for a constant c.
Bounds are achievable in polynomial time!

Line graph: $G = L(H)$



Colouring line graphs: $G = L(H)$

For a multigraph H, $\chi'(H) := \chi(L(H))$.

Goldberg-Seymour Conjecture (weakened)

For any line graph G, $\quad \chi(G) \le \chi_f(G) + 1$.

Theorem (Nishizeki, Kashiwagi '90)

For any H, $\quad \chi'(H) \le \max\{\lfloor 1.1\Delta(H) + 0.8 \rfloor, \lceil \chi'_f(H) \rceil\}$.

Theorem (Kahn '96)

For any H, $\quad \chi'(H) = (1 + o(1))\chi'_f(H)$.

Thus for line graphs, χ is close to χ_f.

Two natural generalizations

Definition

G is quasi-line if every vertex is bisimplicial.

(quasi-line = cobipartite neighbourhoods)

Definition

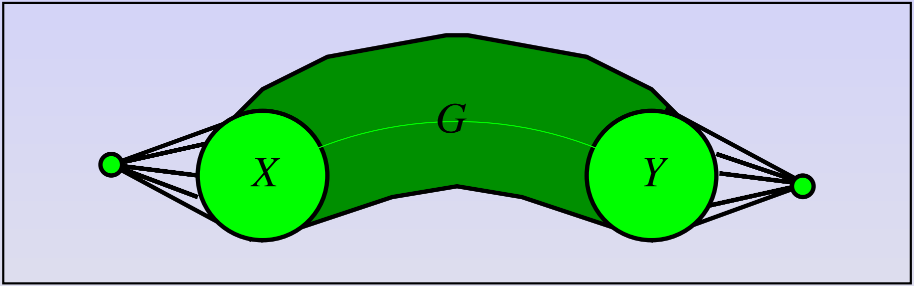
G is claw-free if it contains no induced $K_{1,3}$.

(claw-free = neighbourhoods have $\alpha \leq 2$)

line $\subset$ quasi-line $\subset$ claw-free

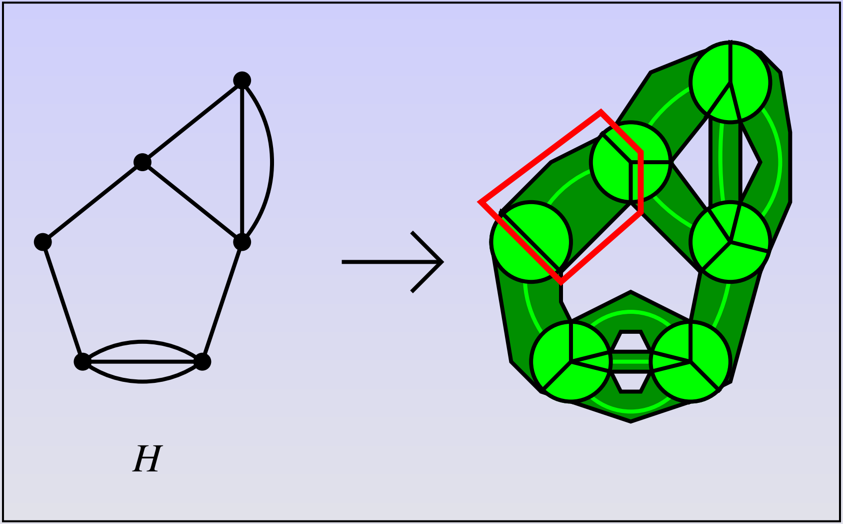
Strips

- A strip is like a claw-free graph with two “endpoints”.



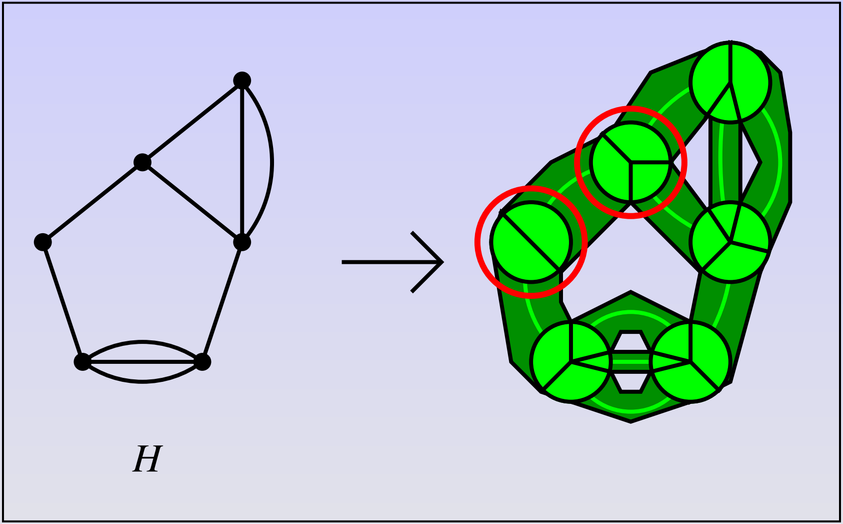
- Endpoints = “end-cliques” = neighbourhoods of two simplicial vertices, which we delete.

Compositions of strips



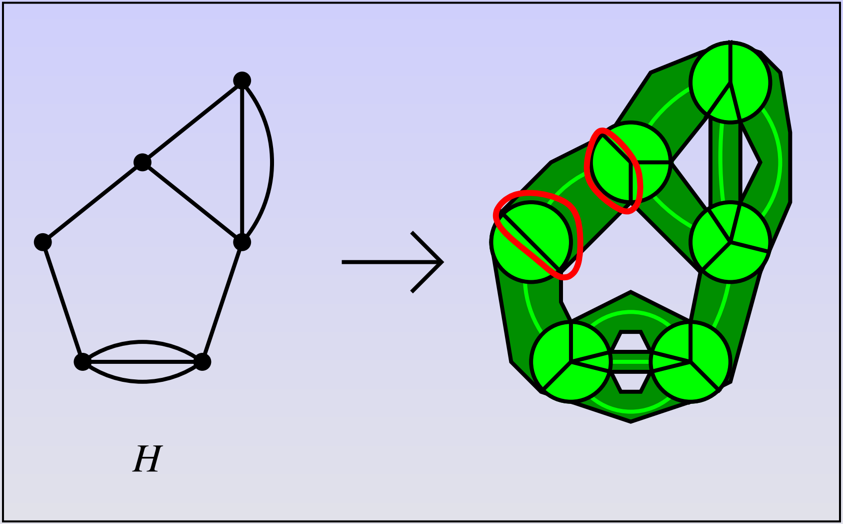
We replace each edge in H with a strip.

Compositions of strips



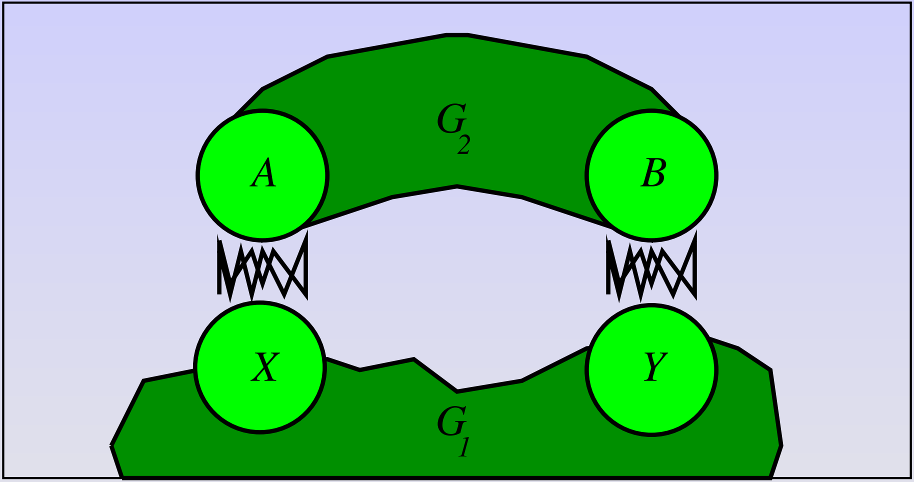
Vertices of H become hub cliques.

Compositions of strips



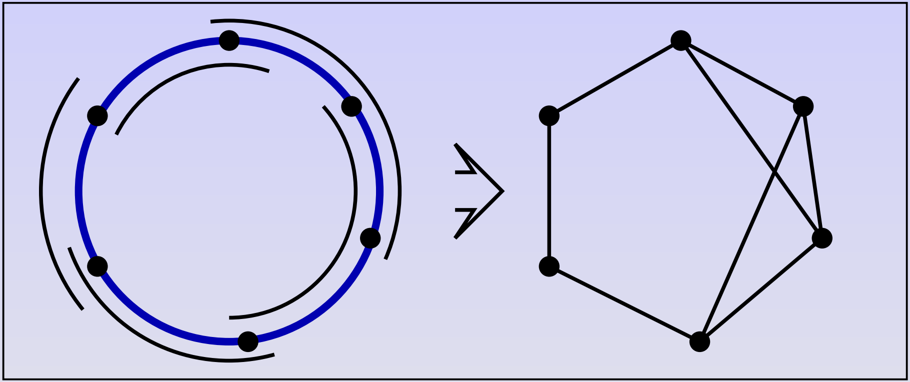
Each hub clique is made up of end-cliques.

Decomposition = 2-join



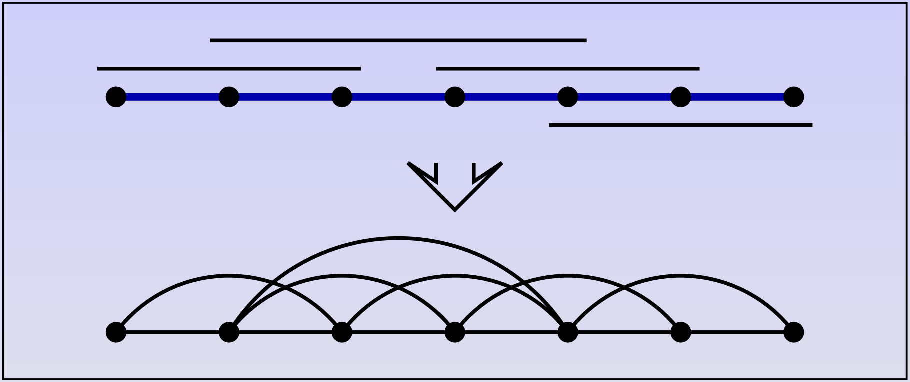
X and Y are the two end-cliques of a claw-free graph.

Circular Interval Graphs



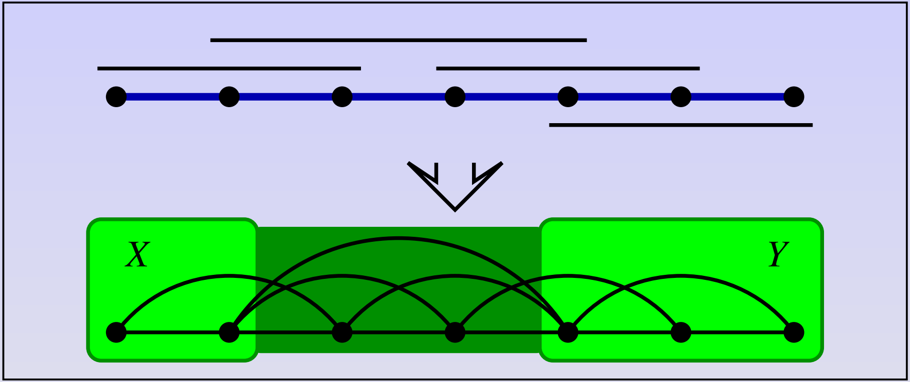
- Points on a circle $\Rightarrow$ vertices.
- Intervals represent cliques.

Linear Interval Graphs



- Points on a line $\Rightarrow$ vertices.
- Intervals represent cliques.

Linear Interval Strips



- End-cliques are at the left and right.
- End-cliques can overlap or even be equal.

The structure of quasi-line graphs

Theorem (Chudnovsky and Seymour)

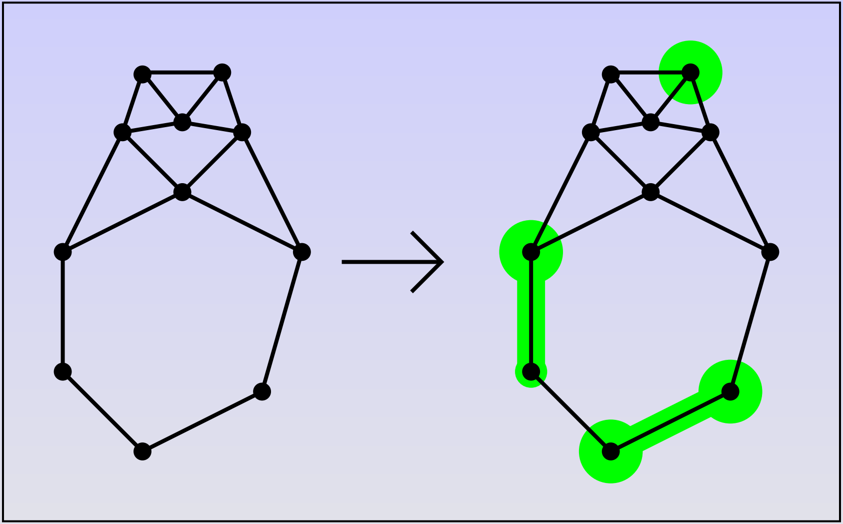
Let G be a quasi-line graph. One of the following holds:

- *G is a circular interval graph.*
- *G is a composition of linear interval strips.*
- *G contains a homogeneous pair of cliques.*

Now we need to deal with homogeneous pairs of cliques.

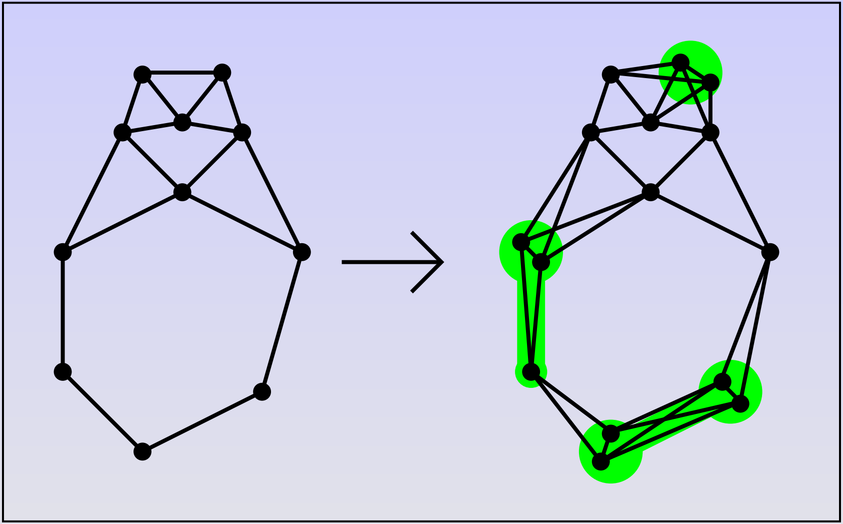
They arise mostly as a result of *thickenings* (vertex expansion)

Thickenings



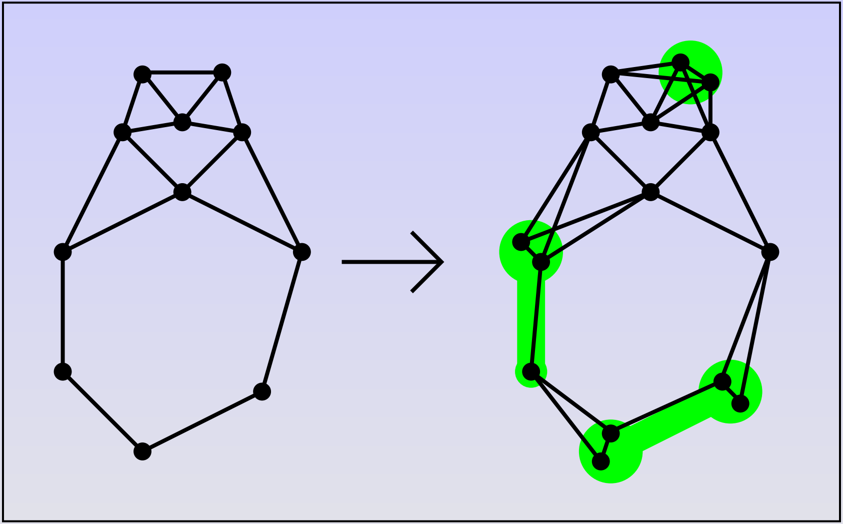
Choose a subset of vertices and a *claw-neutral* matching.

Thickenings



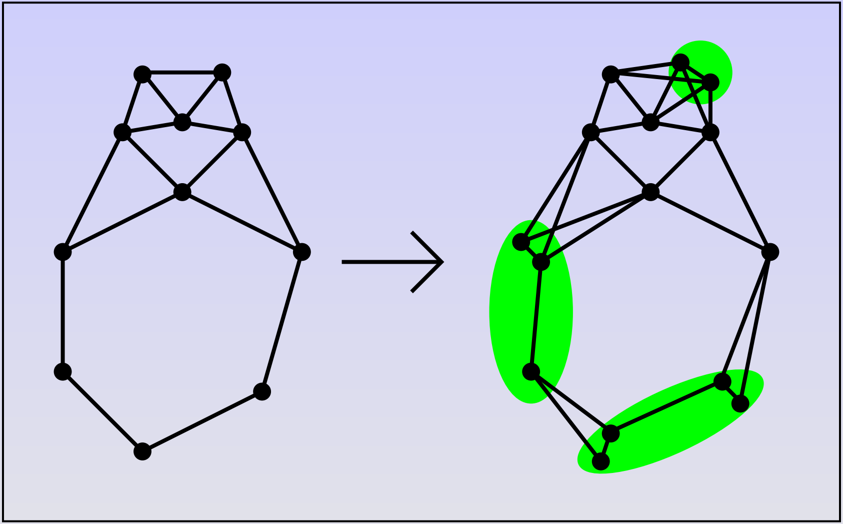
Blow up every vertex into a clique of *twins*.

Thickenings



Remove edges from special homogeneous pairs.

Thickenings



We get homogeneous cliques and homogeneous pairs of cliques.

Thickening

- Thickenings generalize *augmentations*.
- Quasi-line structure theorem generalizes *Berge quasi-line* structure theorem.

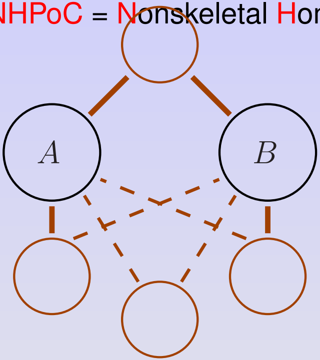
Theorem (Chudnovsky and Seymour)

Let G be a quasi-line graph. One of the following holds:

- *G is a circular interval graph.*
- *G is a composition of linear interval strips.*
- *G contains a nonskeletal homogeneous pair of cliques.*

Homogeneous pairs

NHPoC = Nonskeletal Homogeneous Pair of Cliques (A, B):



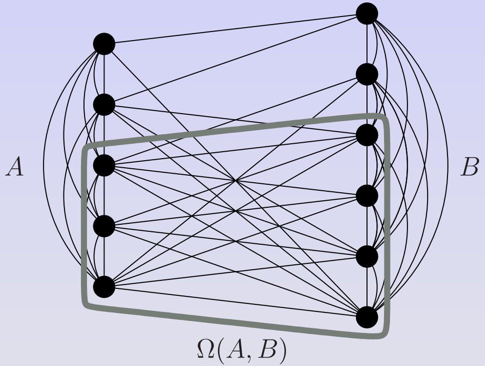
Nonskeletal:
$G[A \cup B]$ has a maximum clique not containing all edges between A and B.

Lemma

If G contains an NHPoC, we can remove edges without lowering $\chi(G)$ or $\chi_f(G)$.

Reducing homogeneous pairs

We can remove edges without changing ω, χ_f, or χ.



. . . until the structure is very simple!

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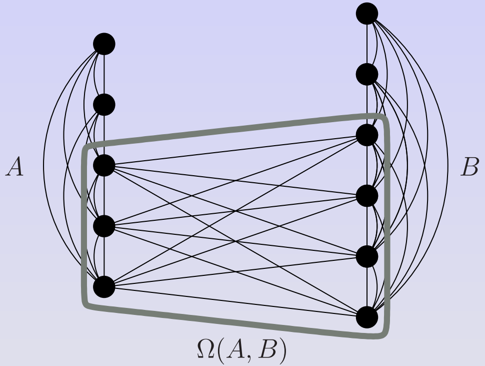

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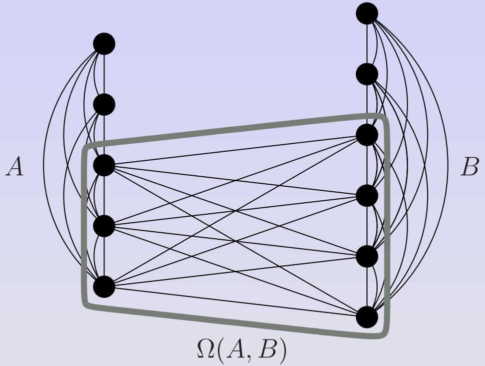
Skeletal homogeneous pairs

The reduced homogeneous pair is skeletal.



A skeletal graph contains no nonskeletal homogeneous pair.

The structure of skeletal quasi-line graphs

Theorem

Let G be a skeletal quasi-line graph. One of the following holds:

- *G is a circular interval graph.*
- *G is a composition of linear interval strips.*

Key Idea

Because of our reductions, we need only consider skeletal graphs.

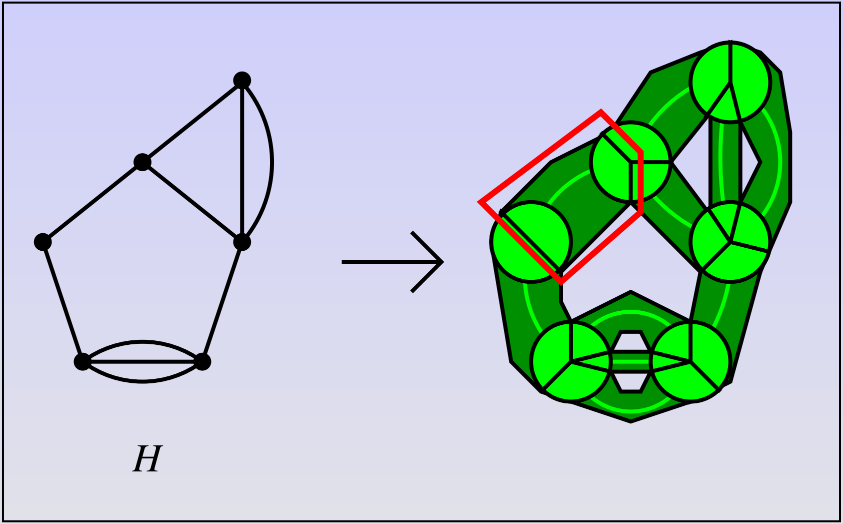
We get circular interval graphs for free.

Theorem (Niessen, Kind '00)

For any circular interval graph, $\chi = \lceil \chi_f \rceil$.

We also use this result to colour many auxiliary graphs.

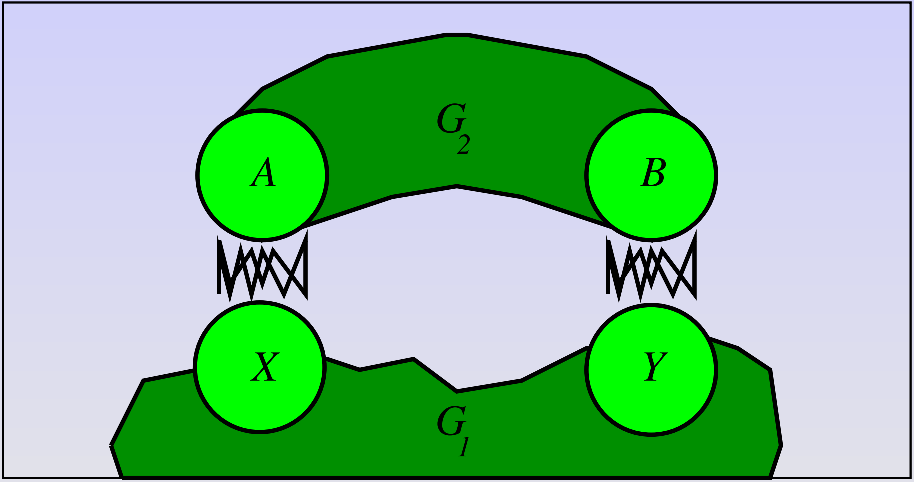
Compositions of strips



We contract the strips to find a colouring of the end-cliques.

Contracting with a fractional colouring

Fractionally colour the graph



Replace G_2 with a clique based on the colouring.

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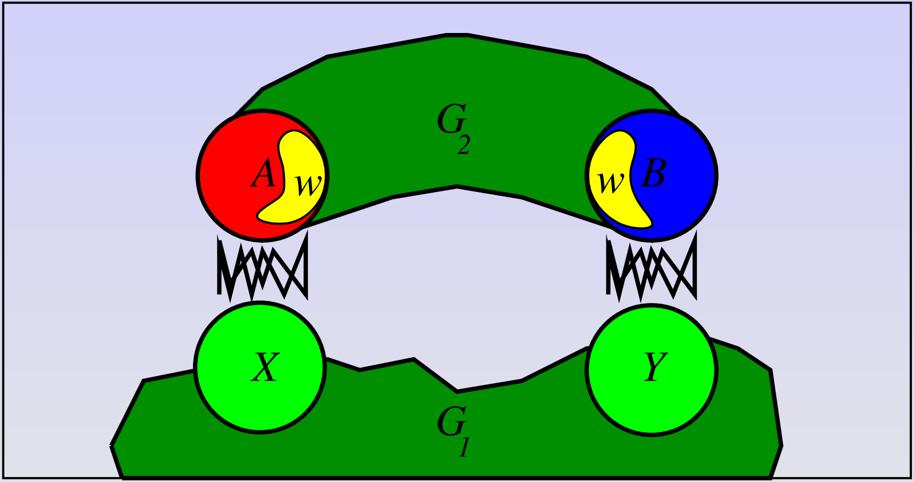

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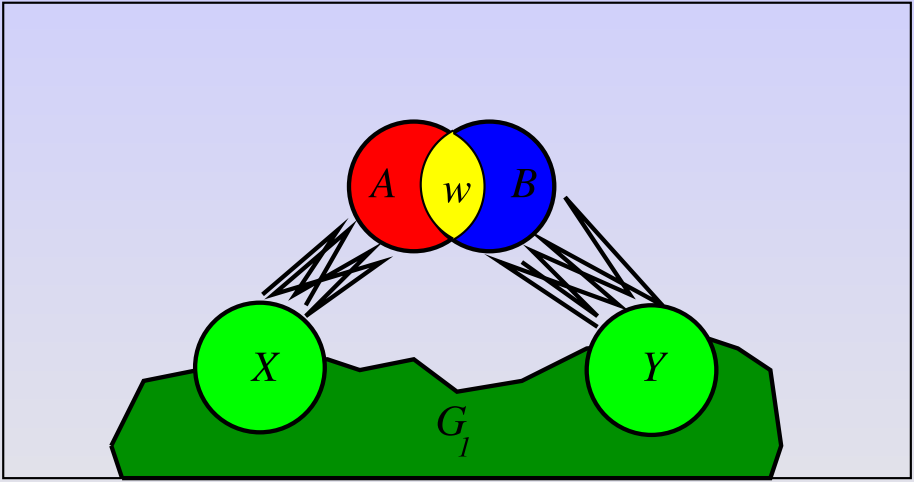

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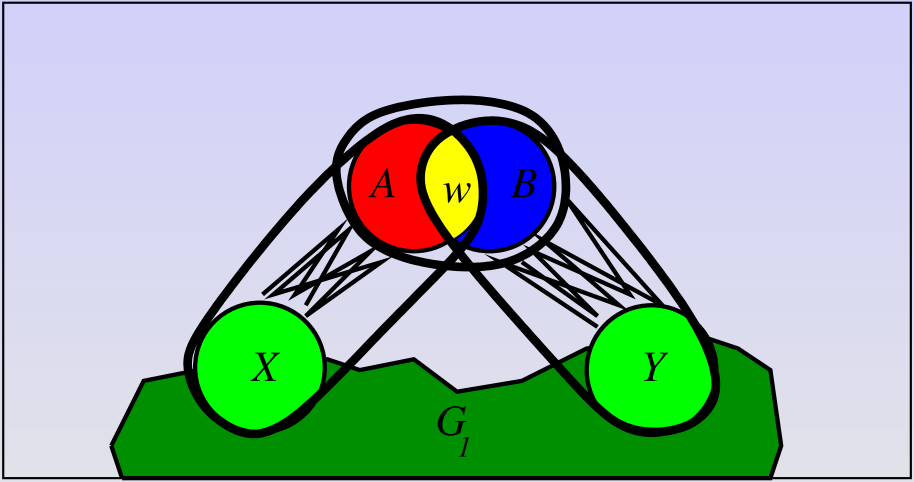

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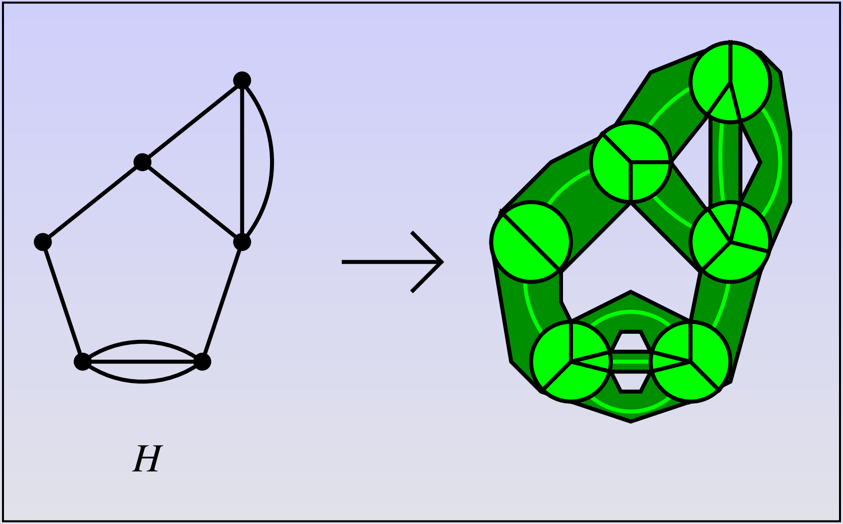
Create a line graph from G



The contractions result in a line graph.

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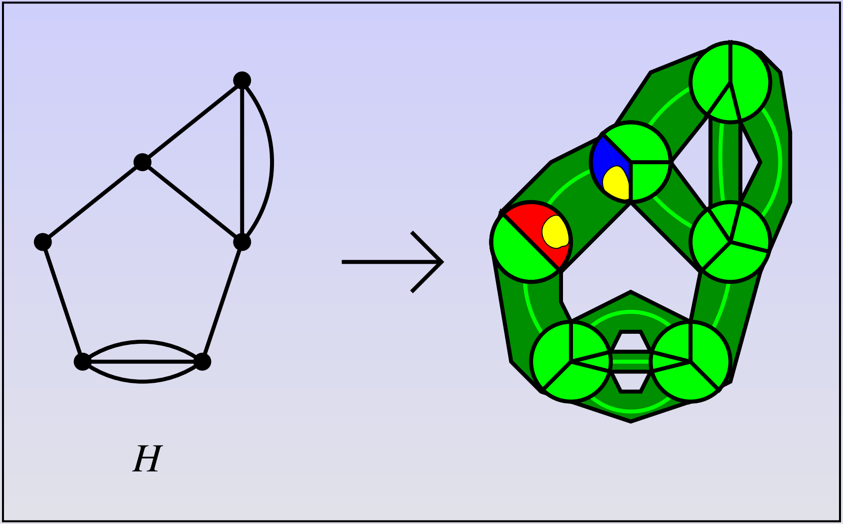

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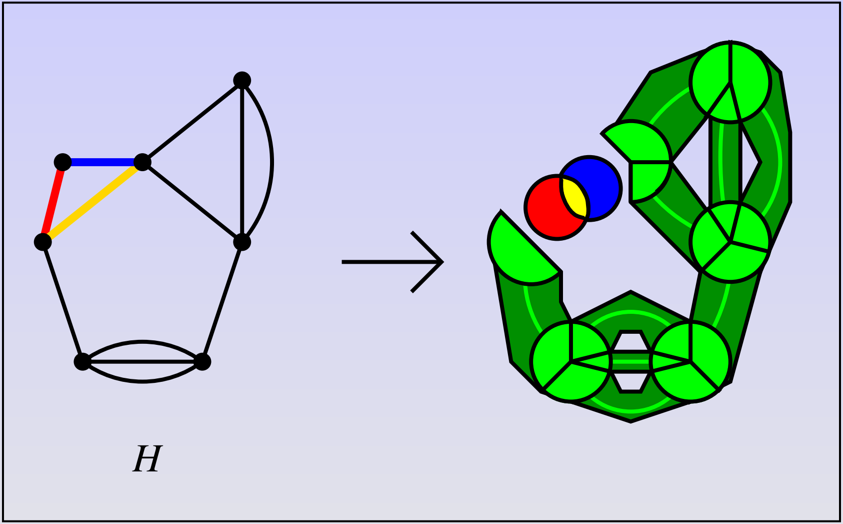

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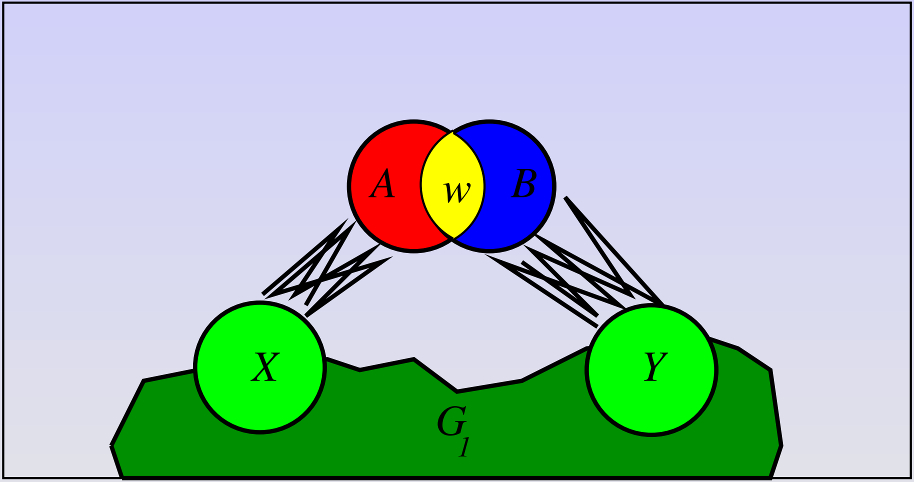
Partially colour G

We get a colouring of the ends of the strips.



Now we fill in the rest.

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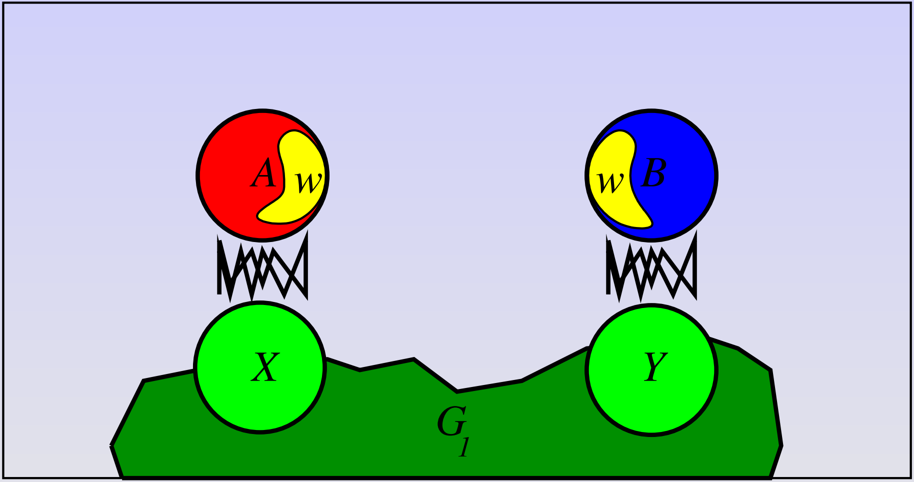

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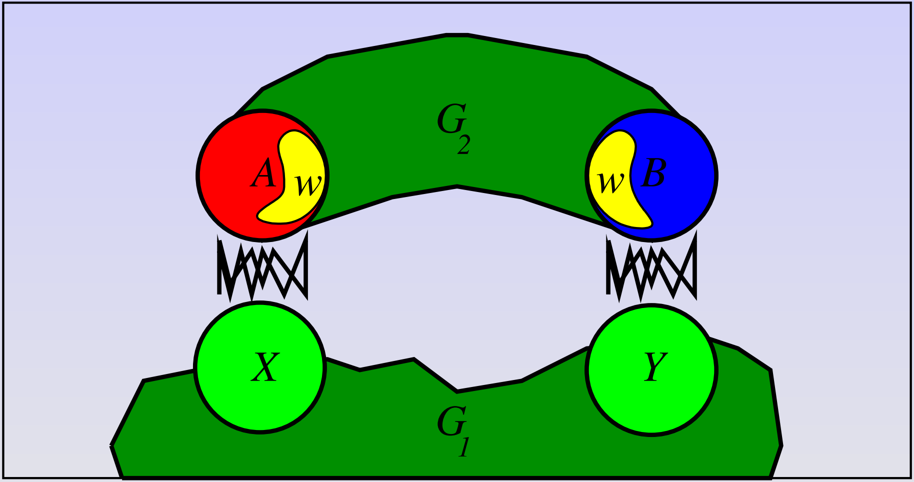

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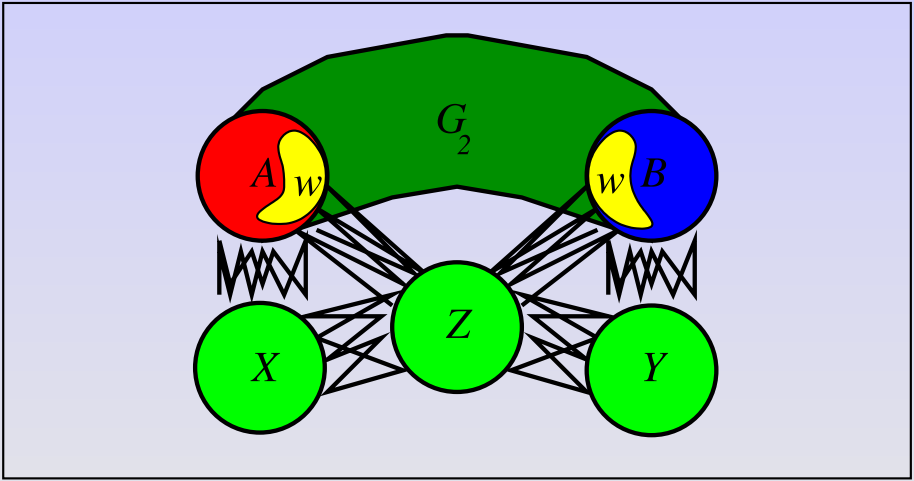
Emulating fractional colourings

We need the right overlap each strip.



We colour an auxiliary circular interval graph.

Extending a theorem

Theorem (Sanders, Steurer)

For any line graph G, $\chi(G) \leq \chi_f(G) + \sqrt{\frac{9}{2}\chi_f(G)}$.

Theorem (K., Reed)

For any quasi-line graph G, $\chi(G) \leq \chi_f(G) + 3\sqrt{\chi_f(G)}$.

These bounds are achievable in polynomial time.

Example

Let G be k copies of C_5 joined together. Then $\chi(G) = 3k$. $\chi_f = \frac{5}{2}k$.

Conjecture

For any claw-free graph G, $\chi(G) \leq \lceil \frac{6}{5}\chi_f(G) \rceil$.

But what if we insist $\alpha(G) \geq 4$? (G must be connected)

Claw-free graphs

Theorem (K., Reed)

For any claw-free graph G with $\alpha(G) \geq 4$,

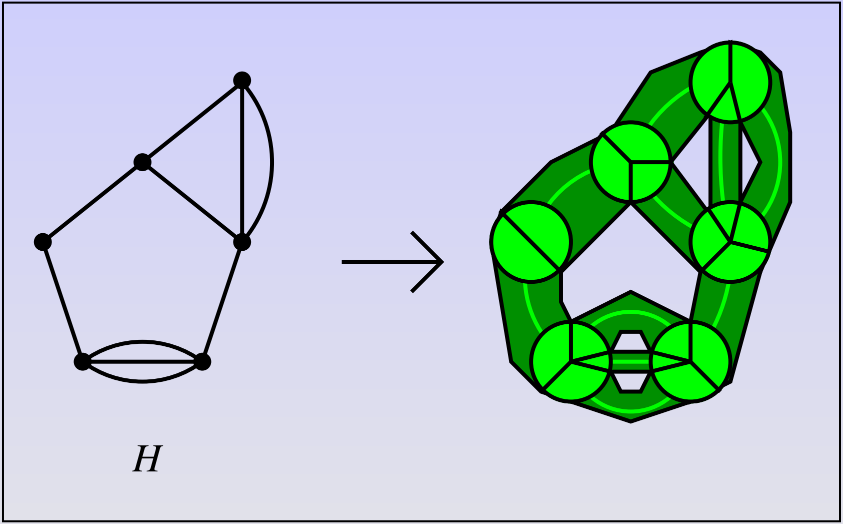
$$\chi(G) \leq \chi_f(G) + c\sqrt{\chi_f(G)}$$

Theorem (Chudnovsky, Seymour)

For G, a skeletal claw-free graph with $\alpha \geq 4$,

- *G is a composition of strips (5 types), or*
- *G is a circular interval graph, or*
- *G admits a 1-join, one side of which is “simple”.*

Why do we care about 1-joins?



1-joins arise from pendant edges in H.

Further applications of the structure theorem

More results on claw-free graphs:

1. (K., Reed) $\chi \leq \lceil \frac{1}{2}(\Delta + 1 + \omega) \rceil$.
2. (Oriolo, Pietropaoli, Stauffer) New algorithm for MWSS problem.
3. (Chudnovsky, Fradkin) $\frac{3}{2}$-approximation of Hadwiger's Conjecture.

And others.