# $K_{s,t}$ minors in graphs

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Based on joint work with N. Prince

# Definitions

A graph $H$ is a minor of a graph $G$
if $H$ can be obtained from $G$ by a sequence of contracting edges and deleting vertices and/or edges.

In this case, we also say that $G$ has an $H$ minor.

A graph is $k$-chromatic if its chromatic number is exactly $k$.

# Conjectures

Conjecture 1. [Hadwiger] Every $k$-chromatic graph has a $K_k$ minor.

Conjecture 2. [Woodall–Seymour] Every $(s+t)$-chromatic graph has a $K_{s,t}$ minor.

Conjecture 3. [Woodall] Every graph with choosability $(s+t)$ has a $K_{s,t}$ minor.

## Another approach

Let $D(H)$ denote the minimum $D$ such that every graph $G$ with average degree at least $D$ has an $H$ minor.

[Kostochka–Thomason]: $D(K_k) = \Theta(k\sqrt{\log k})$.

Myers and Thomason estimated $D(H)$ for most graphs $H$. Their approach worked well for dense and balanced graphs. An example of a sparse unbalanced graph is $K_{s,t}$ for a small $s$ and a large $t$.

Graph $M(r, s, t)$ consists of $r$ copies of $K_{s+t-1}$, where each two copies share the same fixed $s - 1$ vertices.

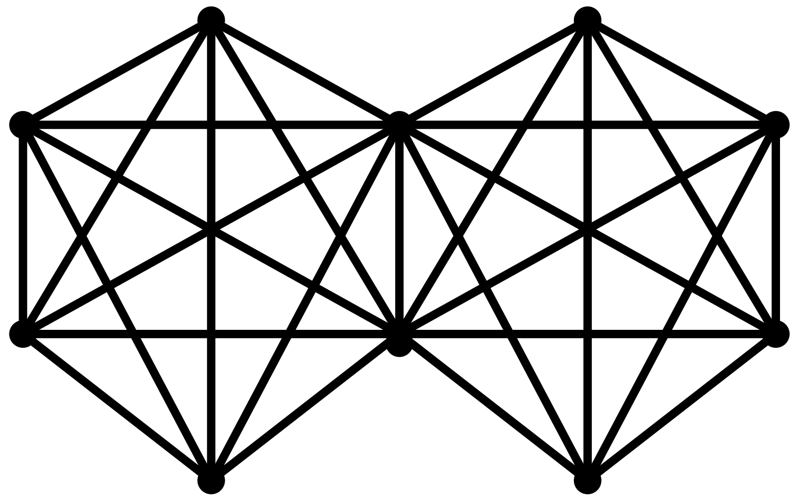


Figure 1: Graph $M(2,3,4)$ has no $K_{3,4}$ minor.

$$e(M(r,s,t)) = \frac{1}{2}(t+2s-3)(n-s+1) + \binom{s-1}{2}. \quad (1)$$

Theorem 1. [Myers] Let $t > 10^{29}$ be a positive integer. Let $G$ be a graph with $n \geq 3$ vertices such that

$$e(G) > \frac{1}{2}(t+1)(n-1). \tag{2}$$

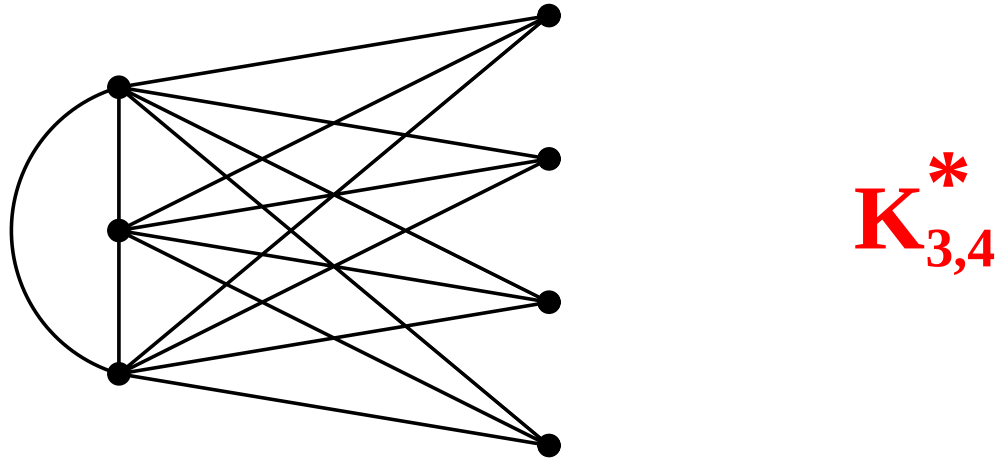
Then $G$ has a $K_{2,t}$ minor.

Theorem 1′. [Chudnovsky–Reed–Seymour] Let $G$ be a graph with $n \geq 3$ vertices satisfying (2). Then $G$ has a $K_{2,t}$ minor.

Corollary. Every $(2+t)$-chromatic graph has a $K_{2,t}$ minor.

Conjecture 3. [Myers] For each fixed $s$, there is $C(s)$ such that for all $t$, $D(K_{s,t}) \leq C(s) \cdot t$.

Let $K^*_{s,t} = K_{s+t} - E(K_t)$.




Myers: Maybe for fixed $s$ and large $t$, $D(K^*_{s,t}) = D(K_{s,t})$?

Theorem 2. [Woodall] Every graph with choosability $(2+t)$ has a $K^*_{2,t}$ minor.

Theorem 3. [Kühn–Osthus] For every $\epsilon > 0$ and every positive integer $s$, there exists a number $t_0 = t_0(s, \epsilon)$ such that for all $t \geq t_0$, every graph of average degree at least $(1+\epsilon)t$ has a $K^*_{s,t}$ minor.

Theorem 4. [A. K.–Prince] Let $s$ and $t$ be positive integers with $t > (240s \log_2 s)^{8s \log_2 s+1}$. Let $G$ be a graph such that

$$e(G) \geq \frac{t+3s}{2}(n(G) - s + 1).$$

Then $G$ has a $K^*_{s,t}$ minor. Furthermore, for $n$ large, there exists a graph $G$ of order $n$ and size at least $\frac{t+3s-5\sqrt{s}}{2}(n - s + 1)$ that has no $K_{s,t}$ minor.

Theorem 5. [A. K.–Prince] Let $t \geq 6500$. Let $G$ be a graph of order $n \geq 3$ with $e(G) > \frac{1}{2}(t+3)(n-2)+1.$ Then $G$ has a $K^*_{3,t}$ minor.

Theorem 6. [A. K.–Prince] Let $s \geq 4$ and $t > 10s$ be such that $s+t$ is odd. Then for infinitely many $n > s+t$, there exists a graph $G(n,s,t)$ of order $n$ with

$$e(G(n,s,t)) > \frac{1}{2}(t+2s-2-\frac{2s}{t})(n-s+1)+\binom{s-1}{2}$$

that has no $K^*_{s,t}$-minor.

Corollary. [Seymour] Let $t \geq 6500$. Then every $(3+t)$-chromatic graph has a $K_{3,t}$ minor.

Theorem 7. [A. K.] Let $s$ and $t$ be positive integers such that

$$t > t_0(s) := \max\{4^{15s^2+s}, (240s\log_2 s)^{8s\log_2 s+1}\}. \quad (3)$$

Then every $(s+t)$-chromatic graph has a $K^*_{s,t}$ minor.

Remark. For every $s, t \geq 3$, there are infinitely many $(s+t)$-critical graphs that do not have $K^*_{s,t+1}$ minors.

# Proof ingredients

Theorem 8. [Gallai] Let $k \geq 3$ and $G$ be a $k$-critical graph. If $|V(G)| \leq 2k-2$, then $G$ has a spanning complete bipartite subgraph.

Suppose that Theorem 7 is proved for all $s' < s$ and $t' > t_0(s')$. Let $G_0$ be a minimum w.r.t. $|V(G)| + |E(G)|$ counter-example for $s$ and some $t > t_0(s)$. Then $G_0$ is $(s+t)$-critical. By Theorem 8, $|V(G_0)| \geq 2(s+t)-1$.

Lemma 1. [Seymour] Let $k \geq 0$. If $v \in V(G_0)$ and $d(v) = s+t-1+k$, then $\alpha(G_0[N(v)]) \leq k+1$.

# Proof ingredients

Lemma 2. If $t \geq 4^{x+s}$, then the connectivity of $G_0$ is greater than $x$.

Lemma 3. If $v \in V(G_0)$ and $d(v) = s + t - 1 + k$, then there exists a subset $Y(v) \subseteq N(v)$ such that $\delta(G_0[Y(v) \cup \{v\}]) \geq \frac{t}{k+1}$.

Lemma 4. Let $t > t_0(s)$. Let $G$ be a $15s^2$-connected graph. Suppose that $G$ contains a vertex subset $U$ with

$$t + 700s^3 \ln t \leq |U| \leq 3t$$

such that $\delta(G[U]) \geq t/(4s + 1)$. Then $G$ has a $K^*_{s,t}$-minor.