$K_{s,t}$ minors in graphs

Alexandr Kostochka

Dept of Mathematics, University of Illinois at Urbana-Champaign

Based on joint work with N. Prince
Definitions

A graph $H$ is a minor of a graph $G$ if $H$ can be obtained from $G$ by a sequence of contracting edges and deleting vertices and/or edges.

In this case, we also say that $G$ has an $H$ minor.

A graph is $k$-chromatic if its chromatic number is exactly $k$. 
Conjectures

Conjecture 1. [Hadwiger] Every $k$-chromatic graph has a $K_k$ minor.

Conjecture 2. [Woodall–Seymour] Every $(s + t)$-chromatic graph has a $K_{s,t}$ minor.

Conjecture 3. [Woodall] Every graph with choosability $(s + t)$ has a $K_{s,t}$ minor.
Another approach

Let $D(H)$ denote the minimum $D$ such that every graph $G$ with average degree at least $D$ has an $H$ minor.

[Kostochka–Thomason]: $D(K_k) = \Theta(k\sqrt{\log k})$.

Myers and Thomason estimated $D(H)$ for most graphs $H$. Their approach worked well for dense and balanced graphs. An example of a sparse unbalanced graph is $K_{s,t}$ for a small $s$ and a large $t$.

Graph $M(r, s, t)$ consists of $r$ copies of $K_{s+t-1}$, where each two copies share the same fixed $s-1$ vertices.
Figure 1: Graph $M(2, 3, 4)$ has no $K_{3,4}$ minor.

$$e(M(r, s, t)) = \frac{1}{2}(t + 2s - 3)(n - s + 1) + \left(\frac{s - 1}{2}\right).$$ (1)
Theorem 1. [Myers] Let \( t > 10^{29} \) be a positive integer. Let \( G \) be a graph with \( n \geq 3 \) vertices such that

\[
e(G) > \frac{1}{2}(t + 1)(n - 1).
\]  

(2)

Then \( G \) has a \( K_{2,t} \) minor.

Theorem 1’. [Chudnovsky–Reed–Seymour] Let \( G \) be a graph with \( n \geq 3 \) vertices satisfying (2). Then \( G \) has a \( K_{2,t} \) minor.

Corollary. Every \((2 + t)\)-chromatic graph has a \( K_{2,t} \) minor.

Conjecture 3. [Myers] For each fixed \( s \), there is \( C(s) \) such that for all \( t \), \( D(K_{s,t}) \leq C(s) \cdot t \).
Let $K_{s,t}^* = K_{s+t} - E(K_t)$.

Myers: Maybe for fixed $s$ and large $t$, $D(K_{s,t}^*) = D(K_{s,t})$?

**Theorem 2.** [Woodall] Every graph with choosability $(2 + t)$ has a $K_{2,t}^*$ minor.
Theorem 3. [Kühn–Osthus] For every $\epsilon > 0$ and every positive integer $s$, there exists a number $t_0 = t_0(s, \epsilon)$ such that for all $t \geq t_0$, every graph of average degree at least $(1 + \epsilon)t$ has a $K_{s,t}^*$ minor.

Theorem 4. [A. K.–Prince] Let $s$ and $t$ be positive integers with $t > (240s \log_2 s)^{8s \log_2 s + 1}$. Let $G$ be a graph such that

$$e(G) \geq \frac{t + 3s}{2}(n(G) - s + 1).$$

Then $G$ has a $K_{s,t}^*$ minor. Furthermore, for $n$ large, there exists a graph $G'$ of order $n$ and size at least $\frac{t + 3s - 5\sqrt{s}}{2}(n - s + 1)$ that has no $K_{s,t}^*$ minor.
Theorem 5. [A. K.–Prince] Let $t \geq 6500$. Let $G$ be a graph of order $n \geq 3$ with $e(G) > \frac{1}{2}(t + 3)(n - 2) + 1$. Then $G$ has a $K_{3,t}^*$ minor.

Theorem 6. [A. K.–Prince] Let $s \geq 4$ and $t > 10s$ be such that $s + t$ is odd. Then for infinitely many $n > s + t$, there exists a graph $G(n, s, t)$ of order $n$ with

$$e(G(n, s, t)) > \frac{1}{2}(t + 2s - 2 - \frac{2s}{t})(n - s + 1) + \binom{s - 1}{2}$$

that has no $K_{s,t}^*$-minor.

Corollary. [Seymour] Let $t \geq 6500$. Then every $(3 + t)$-chromatic graph has a $K_{3,t}$ minor.
**Theorem 7.** [A. K.] Let $s$ and $t$ be positive integers such that

\[ t > t_0(s) := \max\{4^{15s^2 + s}, (240s \log_2 s)^{8s \log_2 s + 1}\}. \]  

Then every $(s + t)$-chromatic graph has a $K_{s,t}^*$ minor.

**Remark.** For every $s, t \geq 3$, there are infinitely many $(s + t)$-critical graphs that do not have $K_{s,t+1}^*$ minors.
Proof ingredients

**Theorem 8.** [Gallai] Let $k \geq 3$ and $G$ be a $k$-critical graph. If $|V(G)| \leq 2k - 2$, then $G$ has a spanning complete bipartite subgraph.

Suppose that Theorem 7 is proved for all $s' < s$ and $t' > t_0(s')$. Let $G_0$ be a minimum w.r.t. $|V(G)| + |E(G)|$ counter-example for $s$ and some $t > t_0(s)$. Then $G_0$ is $(s + t)$-critical. By Theorem 8, $|V(G_0)| \geq 2(s + t) - 1$.

**Lemma 1.** [Seymour] Let $k \geq 0$. If $v \in V(G_0)$ and $d(v) = s + t - 1 + k$, then $\alpha(G_0[N(v)]) \leq k + 1$. 
Proof ingredients

**Lemma 2.** If \( t \geq 4^x + s \), then the connectivity of \( G_0 \) is greater than \( x \).

**Lemma 3.** If \( v \in V(G_0) \) and \( d(v) = s + t - 1 + k \), then there exists a subset \( Y(v) \subseteq N(v) \) such that \( \delta(G_0[Y(v) \cup \{v\}]) \geq \frac{t}{k+1} \).

**Lemma 4.** Let \( t > t_0(s) \). Let \( G \) be a \( 15s^2 \)-connected graph. Suppose that \( G \) contains a vertex subset \( U \) with

\[
t + 700s^3 \ln t \leq |U| \leq 3t
\]

such that \( \delta(G[U]) \geq t/(4s + 1) \). Then \( G \) has a \( K_{s,t}^* \)-minor.