The Erdős-Ko-Rado Theorem for Permutations

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Answer must depend on n, for example, if $2k - t \geq n$ any two k-sets will be t-intersecting!

Examples

A 1-intersecting 3-set system on a 6-set:

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The collection of all k-sets that contain a fixed t-set is called a *trivially t-intersecting k-set system.*

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- **Ahlswede and Khachatrian 1997** Gave the maximal t-intersecting k-set system for all values of n.

Ahlswede and Khachatrian

Define the system of k-subsets from $\{1, ..., n\}$

$$\mathcal{F}_i = \{A \in \binom{[n]}{k} \; : \; |A \cap \{1, 2, ..., t + 2i\}| \geq t + i\}$$

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If

$$(k - t + 1)\left(2 + \frac{t-1}{i+1}\right) < n < (k - t + 1)\left(2 + \frac{t-1}{i}\right),$$

then $\mathcal{A}_i$ is the unique (up to a permutation on $\{1, \ldots, n\}$) t-intersecting k-set system with maximal cardinality.

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There are clear candidates for the largest system:

$$\mathcal{F}_{i,j} = \{\sigma \in S_n \text{ with } \sigma(i) = j\}$$

for any $i, j \in \{1, \ldots, n\}$.
These are coset of the stabiliser of a point.

EKR for Permutations

Theorem (Cameron and Ku (2003), Larose and Malvenuto (2003), Godsil and Meagher (2007),)

Let $\mathcal{F}$ be a system of intersecting permutations on n-elements. Then

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- Godsil and Meagher - used an arguement with eigenvectors.

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(Have bound easily, uniqueness is the hard part.)

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  Equitable paritions of permutations that fix 1 and permutations that don't fix 1.

By Hoffman's ratio bound we have

$$\alpha(D(n)) \leq \frac{|V(D(n))|}{1 - \frac{d(n)}{\tau}} = \frac{n!}{1 - \frac{d(n)}{\frac{d(n)}{n-1}}} = (n-1)!$$

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How do we show that the $\mathcal{F}_{i,j}$ are all the maximum independent sets?

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- What is a maximum clique?

Cliques in $D_t(n)$

A set $H \subseteq S_n$ is called *sharply t-transitive* if for any (ordered?) t-set $a_1, a_2, \cdots a_t$ and any other t-set $b_1, b_2, \cdots, b_t$ there exists a unique permutation $\pi \in H$ such that $\pi(a_i) = b_i$.

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Fix a t-set, each $h \in H$ maps it to a unique t-set so $|H| = n(n-1)\cdots(n-t+1)$. If $\sigma, \pi \in H$ then $\sigma^{-1}\pi$ agree in fewer than t elements (else σ and π would map a t-set to the same t-set)

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Conjecture

For every n the largest t-intersecting permutation system is a $\mathcal{F}_i$ for some i

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If $n \geq 2t + 1$ then $\mathcal{F}_0$ is the largest system.