How to construct (v, k, λ)-designs fast and easily

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Designs

Design

Design is a pair $(X, \mathcal{A})$ where

1. X is a set of elements called **points**, and
2. $\mathcal{A}$ is a multiset of nonempty subsets of X called **blocks**.

$t-(v,k,\lambda)$ - design

Let t, v, k and λ be positive integers such that $v > k \geq 2$. A $t-(v,k,\lambda)-$**design** is a design $(X, \mathcal{A})$ such that the following properties are satisfied :

1. $|X| = v$,
2. each block contains exactly k points, and
3. every set of t distinct points is contained in exactly λ blocks.

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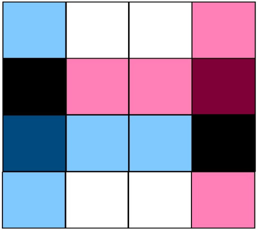
- An example of $(16, 6, 2)$ - design



Automorphism group

Design automorphism

Automorphism of a design $(X, \mathcal{A})$ is a mapping $\pi : X \to X$ so that if $\mathcal{B} \in \mathcal{A}$, then $\pi(\mathcal{B}) \in \mathcal{A}$.

It can be shown that the set of all automorphisms of a $(X, \mathcal{A})$ forms a group G under the operation of composition of mappings.

- **Dihedral group of order 10**

$$G = \langle a, b | a^5 = b^2 = 1, a^b = a^{-1} \rangle$$

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Example (16, 6, 2)

Let G act on $(16,6,2)$ in a following way :

- point orbits $\{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4\}$
- point orbit sizes $\gamma = [1,5,5,5]$

Point orbits

- $\mathcal{O}_1 = \{1\}$
- $\mathcal{O}_2 = \{2,3,4,5,6\}$
- $\mathcal{O}_3 = \{7,8,9,10,11\}$
- $\mathcal{O}_4 = \{12,13,14,15,16\}$

Then we have

$$G = \langle (2,3,4,5,6)(7,8,9,10,11)(12,13,14,15,16),$$

$$(3,6)(4,5)(8,11)(9,10)(13,16)(14,15) \rangle$$

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- $\mathcal{O}_3 = \{7, 8, 9, 10, 11\}$
- $\mathcal{O}_4 = \{12, 13, 14, 15, 16\}$

Then we have

$$G = \langle (2,3,4,5,6)(7,8,9,10,11)(12,13,14,15,16),$$

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$$(3, 6)(4, 5)(8, 11)(9, 10)(13, 16)(14, 15)\rangle$$

Kramer-Mesner

- G = group of permutations of $\{1, 2, \ldots, v\}$
- $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m$ t-orbits of $\{1, \ldots, v\}$
- $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ k-orbits of $\{1, \ldots, v\}$

$$a_{ij} = |\{ B \in \mathcal{B}_j \mid S \subseteq B \}|, \qquad S \in \mathcal{S}_i \quad \text{(does not depend on choice!)}$$

- **Kramer-Mesner matrix** $A_{tk}^G = [a_{ij}]$

- A_{26}^G

	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\mathcal{B}_4$	$\mathcal{B}_5$	$\mathcal{B}_6$	$\mathcal{B}_7$	$\mathcal{B}_8$	…
$\mathcal{S}_1$	1	8	8	4	8	8	4	6	…
$\mathcal{S}_2$	0	2	2	1	0	0	0	4	…
$\mathcal{S}_3$	0	0	0	0	2	2	1	0	…
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\mathcal{S}_{18}$	0	0	0	0	0	0	0	0	…

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$\mathcal{S}_{18}$	0	0	0	0	0	0	0	0	...

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$\mathcal{S}_{18}$	0	0	0	0	0	0	0	0	$\ldots$

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$\mathcal{S}_{18}$	0	0	0	0	0	0	0	0	...

Kramer-Mesner

Theorem

A simple t-(v, k, λ) design with G as a group of automorphisms exists if and only if the system of linear equations

$$A_{tk}^{G} \cdot x = \lambda u_n$$

has a $\{0, 1\}$-solution.

Kramer-Mesner algorithm

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Theorem

If D is a simple t-(v,k,λ) design with G as a group of automorphisms, then D is $s-(v,k,\lambda)$ -design, $1 \leq s \leq t$ and the system of linear equations

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Example 2-(16,6,2)

- calculate 2 - orbits (**18** 2-orbits)
- calculate 6 - orbits (**860** 6-orbits)
- A_{26}^{G} is of size 18×860

We need to solve

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$$\begin{bmatrix} 1 & 8 & 8 & 4 & 8 & 8 & 4 & 6 & 3 & 6 & 6 & 6 & 6 & \dots \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 & 4 & 2 & 4 & 4 & 2 & 2 & \dots \\ 0 & 0 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & \dots \\ 1 & 6 & 6 & 3 & 6 & 6 & 3 & 4 & 2 & 4 & 4 & 4 & 4 & \dots \\ 1 & 6 & 6 & 3 & 6 & 6 & 3 & 2 & 1 & 2 & 2 & 2 & 2 & \dots \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 4 & 2 & 2 & 2 & 2 & 2 & \dots \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 3 & 1 & 2 & 2 & 1 & 1 & \dots \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 3 & 3 & 1 & 1 & \dots \\ 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ \vdots \\ x_{860} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ \vdots \\ 2 \end{bmatrix}$$

Can we reduce this problem ?
YES WE CAN !

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Matrix decomposition

Matrix decomposition

Let A be incidence matrix of a design $(X, \mathcal{A})$.

A **decomposition of** A is any partition $\mathcal{O}_1, ..., \mathcal{O}_r$ of the rows of A (points of $(X, \mathcal{A})$) and a partition $\mathcal{B}_1, ..., \mathcal{B}_s$ of the columns of A (blocks of $(X, \mathcal{A})$).

We say that a decomposition is

- **row-tactical** (**point-tactical**) if the sum of entries of each row in $\mathcal{O}_i, i = 1..., r$ is constant,
- **column-tactical** (**block-tactical**) if the sum of entries of each column in $\mathcal{B}_j, j = 1..., s$ is constant.
- **tactical** if it is both row-tactical and column-tactical.

Tactical decomposition matrix (16, 6, 2)

- incidence matrix $\Rightarrow$ tactical decomposition matrix

1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	1	1	0	0	1	0	0	1
1	0	1	0	0	0	0	0	0	1	1	1	0	1	0	0
1	0	0	1	0	0	1	0	0	0	1	0	1	0	1	0
1	0	0	0	1	0	1	1	0	0	0	0	0	1	0	1
1	0	0	0	0	1	0	1	1	0	0	1	0	0	1	0
0	0	0	1	1	0	1	0	1	1	0	1	0	0	0	0
0	0	0	0	1	1	0	1	0	1	1	0	1	0	0	0
0	1	0	0	0	1	1	0	1	0	1	0	0	1	0	0
0	1	1	0	0	0	1	1	0	1	0	0	0	0	1	0
0	0	1	1	0	0	0	1	1	0	1	0	0	0	0	1
0	0	1	0	0	1	1	0	0	0	0	1	1	0	0	1
0	1	0	1	0	0	0	1	0	0	0	1	1	1	0	0
0	0	1	0	1	0	0	0	1	0	0	0	1	1	1	0
0	0	0	1	0	1	0	0	0	1	0	0	0	1	1	1
0	1	0	0	1	0	0	0	0	0	1	1	0	0	1	1

$\Rightarrow$

1	5	0	0
1	1	2	2
0	2	3	1
0	2	1	3

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$$
\left[\begin{array}{c|ccccc|ccccc|ccccc}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1
\end{array}\right]
\Rightarrow
\left[\begin{array}{cccc}
1 & 5 & 0 & 0 \\
1 & 1 & 2 & 2 \\
0 & 2 & 3 & 1 \\
0 & 2 & 1 & 3
\end{array}\right]
$$

Tactical decomposition matrix

Tactical decomposition matrix

Let $\mathcal{P}$ be tactical decomposition of a design $(X, \mathcal{A})$. Number of points from orbit $\mathcal{O}_i, \forall i$ contained in each block in $\mathcal{B}_j, \forall j$ is constant. Matrix containing these values is called **tactical decomposition matrix**.

Lemma 1.

Let $(X, \mathcal{A})$ be a design, and G an automorphism group of a design $(X, \mathcal{A})$.

Then the point and block orbits of G form a tactical decomposition of $(X, \mathcal{A})$.

Tactical decomposition

Theorem.

Let D be a (v, k, λ) - design with G as group of automorphism.

- Point orbits : $\mathcal{O}_1, \ldots, \mathcal{O}_m$
- Block orbits : $\mathcal{B}_1, \ldots, \mathcal{B}_n$
- $\rho_{ij} = |\{B \in \mathcal{B}_j \,|\, p \in B\}|$, $p \in \mathcal{O}_i$

Then

$$\sum_{j=1}^{n} \rho_{ij} = r, \quad 1 \leq i \leq m$$

$$\sum_{i=1}^{m} \frac{|\mathcal{O}_i|}{|\mathcal{B}_j|} \rho_{ij} = k, \quad 1 \leq j \leq n$$

$$\sum_{j=1}^{n} \frac{|\mathcal{O}_i|}{|\mathcal{B}_j|} \rho_{ij} \rho_{i'j} = \begin{cases} \lambda |\mathcal{O}_i|, & \text{for } i \neq i' \\ \lambda(|\mathcal{O}_i| - 1) + r, & \text{for } i = i' \end{cases}$$

Example (16, 6, 2)

Tactical decomposition matrix candidate

$$T = \begin{bmatrix} 1 & 5 & 0 & 0 \\ 1 & 1 & 2 & 2 \\ 0 & 2 & 3 & 1 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

- Computed with the program *orbmat5qd* developed by V. Krčadinac

Example (16, 6, 2)

Tactical decomposition matrix candidate

$$T = \begin{array}{c|cccc} \gamma \backslash \beta & 1 & 5 & 5 & 5 \\ \hline 1 & 1 & 5 & 0 & 0 \\ 5 & 1 & 1 & 2 & 2 \\ 5 & 0 & 2 & 3 & 1 \\ 5 & 0 & 2 & 1 & 3 \end{array}$$

- each point $p \in \mathcal{O}_i$ is contained in exactly T_{ij} blocks $b \in \mathcal{B}_j$.
- each point in $\mathcal{O}_2$ is contained in exactly 2 blocks in $\mathcal{B}_3$

Connection : Kramer-Mesner $\leftrightsquigarrow$ Tactical decomoposition

Tactical decomosition

- T ... each point $p \in \mathcal{O}_i$ is contained in exactly T_{ij} blocks $b \in \mathcal{B}_j$.

$\gamma \backslash \beta$	1	5	5	5
1	1	5	0	0
5	1	1	2	2
5	0	2	3	1
5	0	2	1	3

Kramer-Mesner

- A_{1k}^G ... each point $p \in \mathcal{O}_i$ is contained in exactly a_{ij} blocks $b \in \mathcal{B}_j$

	1	5	5	5	5	5	5	5	5	5	5	5	5	10	10	5	10	...
1	1	5	5	5	5	0	0	0	0	0	0	0	0	10	10	5	10	...
5	1	1	1	1	1	2	2	2	2	2	2	2	2	8	8	4	8	...
5	0	2	2	2	2	3	3	3	3	1	1	1	1	2	2	1	0	...
5	0	2	2	2	2	1	1	1	1	3	3	3	3	0	0	0	2	...

Connection : Kramer-Mesner $\leftrightsquigarrow$ Tactical decomoposition

Tactical decomosition

- T ... each point $p \in \mathcal{O}_i$ is contained in exactly T_{ij} blocks $b \in \mathcal{B}_j$.

$\gamma \backslash \beta$	1	5	5	5
1	1	5	0	0
5	1	1	2	2
5	0	2	3	1
5	0	2	1	3

Kramer-Mesner

- A^G_{1k} ... each point $p \in \mathcal{O}_i$ is contained in exactly a_{ij} blocks $b \in \mathcal{B}_j$

	1	5	5	5	5	5	5	5	5	5	5	5	5	10	10	5	10	...
1	1	5	5	5	5	0	0	0	0	0	0	0	0	10	10	5	10	...
5	1	1	1	1	1	2	2	2	2	2	2	2	2	8	8	4	8	...
5	0	2	2	2	2	3	3	3	3	1	1	1	1	2	2	1	0	...
5	0	2	2	2	2	1	1	1	1	3	3	3	3	0	0	0	2	...

Connection : Kramer-Mesner $\leftrightsquigarrow$ Tactical decomoposition

- **Tactical decomposition**

$T =$

$\gamma \backslash \beta$	1	5	5	5
1	1	5	0	0
5	1	1	2	2
5	0	2	3	1
5	0	2	1	3

- **Reduced** A_{16}^{G}

	1	5	5	5	5	5	5	5	5	5	5	5	5
1	1	5	5	5	5	0	0	0	0	0	0	0	0
5	1	1	1	1	1	2	2	2	2	2	2	2	2
5	0	2	2	2	2	3	3	3	3	1	1	1	1
5	0	2	2	2	2	1	1	1	1	3	3	3	3

Example (16, 6, 2)

- **18** 2-orbits
- **13** 6-orbits
- reduced A_{26}^{G} is of dimension 18×13

Reduction

$$18 \times 860 \quad \rightsquigarrow \quad 18 \times 13$$

+ 4 additional equations

- **Reduced** A_{16}^{G}

	1	5	5	5	5	5	5	5	5	5	5	5	5
1	1	5	5	5	5	0	0	0	0	0	0	0	0
5	1	1	1	1	1	2	2	2	2	2	2	2	2
5	0	2	2	2	2	3	3	3	3	1	1	1	1
5	0	2	2	2	2	1	1	1	1	3	3	3	3

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$+$ 4 additional equations

- **Reduced** A_{16}^G

	1	5	5	5	5	5	5	5	5	5	5	5	5
1	1	5	5	5	5	0	0	0	0	0	0	0	0
5	1	1	1	1	1	2	2	2	2	2	2	2	2
5	0	2	2	2	2	3	3	3	3	1	1	1	1
5	0	2	2	2	2	1	1	1	1	3	3	3	3

Example (16, 6, 2)

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Reduction

$$18 \times 860 \quad \rightsquigarrow \quad 18 \times 13$$

+ 4 additional equations

- **Reduced** A_{16}^G

	1	5	5	5	5	5	5	5	5	5	5	5	5
1	1	5	5	5	5	0	0	0	0	0	0	0	0
5	1	1	1	1	1	2	2	2	2	2	2	2	2
5	0	2	2	2	2	3	3	3	3	1	1	1	1
5	0	2	2	2	2	1	1	1	1	3	3	3	3

Example (16, 6, 2)

- **18** 2-orbits
- **13** 6-orbits
- reduced A^G_{26} is of dimension 18×13

Reduction

$$18 \times 860 \quad \rightsquigarrow \quad 18 \times 13$$

+ 4 additional equations

- **Reduced** A^G_{16}

	1	5	5	5	5	5	5	5	5	5	5	5	5
1	1	5	5	5	5	0	0	0	0	0	0	0	0
5	1	1	1	1	1	2	2	2	2	2	2	2	2
5	0	2	2	2	2	3	3	3	3	1	1	1	1
5	0	2	2	2	2	1	1	1	1	3	3	3	3

Example (16, 6, 2)

- **18** 2-orbits
- **13** 6-orbits
- reduced A_{26}^{G} is of dimension 18×13

Reduction

$$18 \times 860 \quad \rightsquigarrow \quad 18 \times 13$$

+ 4 additional equations

- **Reduced** A_{16}^{G}

	1	5	5	5	5	5	5	5	5	5	5	5	5
1	1	5	5	5	5	0	0	0	0	0	0	0	0
5	1	1	1	1	1	2	2	2	2	2	2	2	2
5	0	2	2	2	2	3	3	3	3	1	1	1	1
5	0	2	2	2	2	1	1	1	1	3	3	3	3

Reduced Kramer-Mesner matrix A_{26}^{G}

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 \\
0 & 0 & 0 & 1 & 1 & 2 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 2 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 1 & 2 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \\ x_{13}
\end{bmatrix}
=
\begin{bmatrix}
2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2
\end{bmatrix}
$$

Example (16, 6, 2)

$$x = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}$$

Examples of reduction

v	k	λ	Reduction		
16	6	2	18×860	$\rightsquigarrow$	18×13
7	3	1	7×13	$\rightsquigarrow$	7×7
13	5	5	26×430	$\rightsquigarrow$	26×118
13	3	1	26×99	$\rightsquigarrow$	26×42
22	7	6	15×8167	$\rightsquigarrow$	15×457
15	5	10	7×152	$\rightsquigarrow$	7×88
14	4	6	5×54	$\rightsquigarrow$	5×26
15	5	4	7×52	$\rightsquigarrow$	7×42

Results

v	k	λ	Handbook	Found
14	4	6	> 4	> 1091
14	4	15	> 1	> 63974
15	5	8	> 104	> 192
15	5	10	> 1	> 296717
15	6	10	> 118	> 3727
21	6	4	> 1	> 1351
21	6	5	> 1	> 5328
21	6	6	> 1	> 16977
22	6	5	> 3	> 4897
22	7	6	> 1	> 28