

# TP<sub>2</sub> Completion Problem

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## Outline

- $TP_2$  Completion
  - ☞  $TP_2$  Completion Problem
- $TP_2$  Partial Order and Bruhat Partial Order
  - ☞  $TP_2 = \text{Bruhat}$

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- **$TP_2$  Matrix:** An  $m \times n$  matrix in which every minor of size at most 2 is positive.
- **Partial  $TP_2$  Matrix:** An  $m \times n$  matrix in which some entries are specified and the remaining entries are unspecified, and every minor of specified entries of size at most 2 is positive.

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- *Example:*

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$$\begin{pmatrix} 1 & x & 1 \\ y & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}$$

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$$\mathcal{P} = \begin{pmatrix} a_{11} & x_{12} & a_{13} \\ x_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Partial  $TP_2$  matrix  $\mathcal{P}$  is  $TP_2$  completable iff  $a_{11}a_{23}a_{32} > a_{13}a_{22}a_{31}$ .

## Bruhat Order on Permutations

- *Upward Transposition:* for  $\pi \in S_n$ , a transposition of  $i$  and  $j$  when  $i < j$  and  $\pi(i) < \pi(j)$ .
- *Bruhat Order:* Suppose  $\pi \neq \sigma \in S_n$ . If  $\sigma$  can be obtained from  $\pi$  by a sequence of upward transpositions, then  $\pi$  is said to be less than  $\sigma$  in the *Bruhat partial order*. We denote this by  $\pi <_{Br} \sigma$ .
- *Example:*

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## Notations

- Let  $\pi$  be a permutation on  $I \subseteq \{1, 2, \dots, n\}$ , then  $M_\pi = [m_{ij}] \in M_n$  defined as following

$$m_{ij} = \begin{cases} 1, & \text{if } j = \pi(i) \text{ for } i \in I \\ 0, & \text{otherwise} \end{cases}$$

## Notations

- $M_\pi[p, q]$  : the number of ones in the submatrix of  $M_\pi$  lying in the rows  $1, 2, \dots, p$  and columns  $1, 2, \dots, q$ .

$$M_\pi = \begin{matrix} & 1 & \dots & q \\ \begin{matrix} 1 \\ \vdots \\ p \end{matrix} & \begin{pmatrix} * & \dots & * \\ * & \dots & * \\ & \vdots & \\ * & \dots & * \end{pmatrix} \end{matrix}$$

## Lemma (known)

For two permutations  $\sigma \neq \pi \in S_n$ ,  $\pi <_{Br} \sigma$  if and only if  $M_\pi[i, j] \geq M_\sigma[i, j]$  for all  $i, j \in \{1, 2, \dots, n\}$ .

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- Given a matrix  $A \in M_n$  and a permutation  $\pi \in S_n$ , let

$$A_\pi = \prod_{i=1}^n a_{i\pi(i)}.$$

- *TP<sub>2</sub> Partial Order*: If  $\pi, \sigma \in S_n$  are such that

$$A_\pi < A_\sigma, \quad \forall A \in \text{TP}_2(n, n),$$

then  $\pi$  is less than  $\sigma$  in the *TP<sub>2</sub> partial order*. We denote this as

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## Theorem

For  $\pi, \sigma \in S_n$ ,  $\pi \neq \sigma$ , we have  $\sigma <_{Br} \pi$  if and only if  $\pi <_{TP_2} \sigma$ .

- $\pi <_{TP_2} \sigma \Rightarrow \sigma <_{Br} \pi$

$$K = \left[ \begin{array}{c|c} 2J & J \\ \hline J & J \end{array} \right]$$

$$2^{M_\sigma[p,q]} = \prod_{l=1}^n k_{l\sigma(l)} < \prod_{l=1}^n k_{l\pi(l)} = 2^{M_\pi[p,q]}$$

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A matrix  $A > 0$  is  $TP_2$  if and only if  $A_\pi < A_\sigma$  whenever  $\sigma <_{Br} \pi$ .

- $(\sigma <_{Br} \pi \Rightarrow A_\pi < A_\sigma) \Rightarrow A$  is  $TP_2$

$$A = \begin{bmatrix} & \vdots & \vdots & \\ \dots & a_{ij} & a_{i(j+1)} & \dots \\ \dots & a_{i+1} & a_{(i+1)(j+1)} & \dots \\ & \vdots & \vdots & \end{bmatrix}$$

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Thank You!