minors in large $t$-connected graphs

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Complete minors

A graph H is a minor of graph G if H can be obtained from G by repeated contraction of edges and deletion of edges and vertices.

A graph G contains a complete graph $K_t$ on t vertices as a minor if and only if there exist vertex disjoint connected subgraphs $H_1, H_2, ..., H_t$ of G such that for $i \neq j$ there exists an edge from $V(H_i)$ to $V(H_j)$ in G.

**Theorem (Kuratowski, Wagner):** A graph G is non-planar if and only if G contains $K_5$ or $K_{3,3}$ as a minor.
Excluding complete minors

A graph $G$ has no $K_3$ minor if and only if $G$ is a forest.

A graph $G$ has no $K_4$ minor if and only if $G$ is series-parallel.

**Wagner’s theorem (1937):** A graph $G$ has no $K_5$ minor if and only if it can be obtained from planar graphs and $V_8$ by “gluing” them along cliques of size at most 3.
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A graph $G$ has no $K_6$ minor if and only if ???
Classes of graphs with no $K_6$ minor

- Apex graphs (G is apex if $G - v$ is planar for some $v \in V(G)$)
- Double cross graphs
- Hose graphs
- etc.
If $G$ has no $K_t$ minor then the graph obtained from $G$ by adding a single vertex and joining it to vertices of $G$ arbitrarily has no $K_{t+1}$ minor.

In particular, if $G$ contains a set of vertices $X$ such that $|X| \leq t-5$ and $G - X$ is planar then $G$ has no $K_t$ minor.
**Hadwiger’s conjecture**

**Hadwiger’s conjecture:** Every loopless graph with no $K_t$ minor is $(t-1)$-colorable.

Easy for $t \leq 4$.

By Wagner’s theorem Hadwiger’s conjecture for $t=5$ is equivalent to the Four Color Theorem.

**Theorem (Robertson, Seymour, Thomas, 1993):** Hadwiger’s conjecture holds for $t=6$.

**Theorem (Mader, 1968):** Minimal counterexample to Hadwiger’s conjecture, if it exists, is 6-connected.

**Conjecture (Jorgensen, 1994):** Every 6-connected graph with no $K_6$ minor is apex.
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**Theorem (M. DeVos, R. Hegde, K. Kawarabayashi, S. N., R. Thomas, P. Wollan)**: There exists an integer $N>0$ such that every 6-connected graph on at least $n$ vertices with no $K_6$ minor is apex.
Main result

**Theorem (S.N., Robin Thomas):**

There exists an integer $N$ such that every $t$-connected graph $G$ with $|V(G)| \geq N$ and no $K_t$ minor contains a set $X$ of exactly $(t-5)$ vertices such that $G - X$ is planar.
Main result

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Why large?

Theorem (Kostochka, Thomason, 2001): There exist graphs with connectivity $\Theta(t\sqrt{\log t})$ and no $K_t$ minor.

In the proof we are using Graph Structure of Robertson and Seymour:

Every graph with no $K_6$ minor either is tree like or can be almost embedded in a surface.

And Erdos-Posa like argument:

"What happens once might as well not happened at all."

We want to be able to eliminate parts of the graph that have qualities not replicated elsewhere without losing much of the structure.
A graph $G$ is said to be $k$-linked if it has at least $2k$ vertices and for any ordered $k$-tuples $(s_1, s_2, ..., s_k)$ and $(t_1, t_2, ..., t_k)$ of $2k$ distinct vertices in $G$, there exist pairwise vertex disjoint paths $P_1, P_2, ..., P_k$ such that for $i=1,2,...,k$ $P_i$ has ends $s_i$ and $t_i$.

**Conjecture (Thomassen):** *(which we convinced ourselves is true)*

Every sufficiently large $(2k+2)$-connected graph is $k$-linked.
Theorem (S.N., Robin Thomas): There exists a integer $N$ such that every $t$-connected graph $G$ with $|V(G)| \geq N$ and no $K_t$ minor contains a set $X$ of exactly $(t-5)$ vertices such that $G - X$ is planar.

The structure of large $(t-1)$-connected graphs with no $K_t$ minor can be complicated.

There exist graphs with connectivity $\Theta(t\sqrt{\log t})$ and no $K_t$ minor.

The method has other applications.