

# Asymptotics of Smallest Components Sizes in Decomposable Combinatorial Structures of Alg-Log Type

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# Decomposable structures

A **decomposable combinatorial structure** consists of simpler objects called **components** which by themselves can not be further decomposed.

For example, permutations decompose into cycles, graphs into connected components, and polynomials over finite fields into irreducible factors.

The **size** of an object is the number of its elements. For example, a permutation on  $n$  objects, a graph on  $n$  vertices and a monic  $n$ -degree polynomial over a finite field have all size  $n$ .

## Related work

- Riddel and Uhlenbeck (1953):  
exponential formula in chemistry and physics;
- Foata & Schützenberger (1970):  
exponential formula for labelled objects;
- Bender & Goldman (1971):  
exponential formula for unlabelled objects;
- Joyal (1981) and Bergeron, Labelle & Leroux (1994):  
theory of species;
- Wilf (1990): exponential formula with cards, hands and decks;
- Flajolet (1990's):  
the symbolic method and exp-log/alg-log classes;
- Arratia, Barbour & Tavaré (1990's): probabilistic approach;

## Related work (cont.)

- Labelle & Leroux (1996), Cameron (1997):  $A(C(z))$ ;
- Goncharov (1942, 1962) and Shepp & Lloyd (1966): cycle distribution in permutations;
- Stepanov (1969): distributions in random mappings;
- Knuth & Trabb-Pardo (1976): permutations and numbers;
- Flajolet & Soria (1990, 1993): number of components in exp-log and alg-log classes;
- Gourdon (1996): largest size of components in exp-log and alg-log;
- Panario & Richmond (2001): smallest size of components in exp-log.

# Factorization pattern

The **factorization pattern** of a decomposable combinatorial object of size  $n$  is an  $n$ -tuple  $\alpha_1, \alpha_2, \dots, \alpha_n$  such that the object has  $\alpha_i$  components of size  $i$ , and  $\sum_{i=1}^n i\alpha_i = n$ .

The **restricted pattern** of a decomposable object is a mapping  $S : J \mapsto \mathbb{N}$ , where  $J$  is a set of components' sizes,  $\mathbb{N}$  is the set of nonnegative integers, and  $S(j)$  is the number of components of size  $j$ . If we do not have a restricted pattern,  $J$  is the empty set.

Notation:  $S = \left[ \prod_{j \in J} j^{S(j)} \right]$ .

We derive asymptotic expressions for the probability that a random decomposable object has a given restricted pattern and a restricted size on its  **$r$ th smallest component**.

# Log and alg-log components

We turn the set of all structures of size  $n$  into a probability space using the uniform distribution. Let  $X_n(r)$  be the size of the  $r$ th smallest component in a random structure of size  $n$ . We derive results about the limiting distribution of  $X_n(r)$  when the component generating function  $C(z)$  is of [algebraic-logarithmic](#) type. We also study the asymptotic properties of structures with a restricted pattern.

When the component generating function  $C(z)$  is of [logarithmic](#) type, the smallest component size has been studied by Panario and Richmond (2001), Arratia, Barbour and Tavaré (2003), and Dong, Gao and Panario (2007).

# Generating functions

Notation:

$$C(z) = \sum_{k \geq 1} C_k \frac{z^k}{k!},$$

$$C(z) = \sum_{k \geq 1} C_k z^k,$$

$$C(z; J) = \sum_{j \in J} C_j \frac{z^j}{j!},$$

$$C(z; J) = \sum_{j \in J} C_j z^j,$$

$$\mathcal{S} = \begin{cases} \prod_{j \in J} \left( \frac{C_j}{j!} \right)^{S(j)} \frac{1}{S(j)!} & \text{for labeled case,} \\ \prod_{j \in J} \binom{C_j + S(j) - 1}{S(j)} & \text{for unlabeled case.} \end{cases}$$

Let  $m' = \sum_{j \in J} j S(j)$  denote the size of the restricted pattern, and  $m = n - m'$  be the total size of the unrestricted components.

# Generating function with a restricted pattern

**Lemma.** Let  $S: J \mapsto \mathbb{N}$  be a given restricted pattern and  $F(z; S)$  be the generating function of structures with the restricted pattern  $S$ . For the labeled case, we have

$$F(z; S) = \mathcal{S} z^{m'} \exp(C(z) - C(z; J)).$$

For the unlabeled case, we have

$$F(z; S) = \mathcal{S} z^{m'} \exp\left(\sum_{k \geq 1} \frac{C(z^k) - C(z^k; J)}{k}\right).$$

We use  $|J|$  to denote the number of elements in the set  $J$ , and  $\hat{j}$  to denote the maximum element in  $J$ .

# Component generating functions of alg-log type

Let

$$\Delta(\nu, \theta) = \{z : |z| \leq \rho(1 + \nu), z \neq \rho, |\text{Arg}(z - \rho)| \geq \theta\}$$

for some constants  $\rho > 0$ ,  $\nu > 0$ , and  $0 < \theta < \pi/2$ . We say that  $C(z)$  is of **alg-log type** at singularity  $\rho$ , if  $C(z)$  is analytic in  $\Delta(\nu, \theta)$ , and for some constants  $c$  and  $d$ ,

$$C(z) = c + d(1 - z/\rho)^\alpha \left( \ln \frac{1}{1 - z/\rho} \right)^\beta (1 + o(1)),$$

as  $z \rightarrow \rho$  in  $\Delta(\nu, \theta)$ .

We call  $\alpha$  the **algebraic exponent**, and  $\beta$  the **logarithmic exponent**.

The special case  $\alpha = 0$ ,  $\beta = 1$  is the **logarithmic** type.

# Alg-log type with $0 < \alpha < 1$

We first consider the case that the algebraic exponent  $\alpha$  satisfies  $0 < \alpha < 1$ . In this case, as  $z \rightarrow \rho$  in  $\Delta(\rho, \phi)$  and for **labeled structures**, we have

$$F(z, \emptyset) = e^c + de^c(1 - z/\rho)^\alpha \left( \ln \frac{1}{1 - z/\rho} \right)^\beta (1 + o(1)).$$

Flajolet and Odlyzko's singularity analysis implies

$$[z^n]F(z; \emptyset) \sim e^c \frac{d}{\Gamma(-\alpha)} \rho^{-n} n^{-\alpha-1} (\ln n)^\beta,$$

and

$$\frac{C_n}{n!} \sim \frac{d}{\Gamma(-\alpha)} \rho^{-n} n^{-\alpha-1} (\ln n)^\beta.$$

# Labeled structures

**Theorem.** Assume  $0 < \alpha < 1$ ,  $\hat{j} = O(m/\ln m)$ , and

$$\sum_{j \in J} j^{-\alpha} (\ln j)^{\beta} = o(m^{1-\alpha} (\ln m)^{\beta-1}).$$

Then, as  $m \rightarrow \infty$ , and uniformly in all  $S$

$$[z^n]F(z; S) \sim de^c \frac{\mathcal{L}}{\Gamma(-\alpha)} m^{-\alpha-1} (\ln m)^{\beta} \rho^{-m} \exp(-C(\rho; J)). \quad \square$$

A random structure is chosen uniformly at random from a given family of structures:

$$P(S, n) = \frac{[z^n]F(z; S)}{[z^n]F(z; \emptyset)}$$

is the probability that a random decomposable combinatorial structure of size  $n$  has a restricted pattern  $S$ .

## Corollary

**Corollary.** For labeled structures with algebraic exponent  $0 < \alpha < 1$  such that  $\hat{j} = \max\{j; j \in J\} = O(m/\ln m)$ , and

$$\sum_{j \in J} \frac{(\ln j)^\beta}{j^\alpha} = o\left(m^{1-\alpha} (\ln m)^{\beta-1}\right),$$

we have, as  $m \rightarrow \infty$ ,

$$P(S, n) \sim \left(\frac{n}{m}\right)^{1+\alpha} \left(\frac{\ln m}{\ln n}\right)^\beta \prod_{j \in J} \frac{1}{S(j)!} (C_j \rho^j / j!)^{S(j)} e^{-C_j \rho^j / j!}.$$

If, in addition, the restricted pattern  $S = \left[\prod_{j \in J} j^{S(j)}\right]$  also satisfies  $\sum_{j \in J} j S(j) = o(n)$ , that is,  $m \sim n$ , we have as  $n \rightarrow \infty$ ,

$$P(S, n) \sim \prod_{j \in J} P\left(\left[j^{S(j)}\right], n\right) \sim \prod_{j \in J} \frac{1}{S(j)!} (C_j \rho^j / j!)^{S(j)} e^{-C_j \rho^j / j!}.$$

# The $r$ th smallest component size

Let  $X_n^{[r]}$  and  $X_n^{[r]}(S)$  denote the sizes of the  $r$ th smallest component in a random structure of size  $n$  without or with a restricted pattern  $S$ , respectively.

We define the set  $N_k = \{1, 2, \dots, k\}$  where  $k$  is a positive integer and  $d(k) = \sum_{j \in J \cap N_k} S(j)$ . We only need to consider  $d(k) < r$  since  $P(X_n^{[r]}(S) > k) = 0$  when  $d(k) \geq r$ .

**Theorem.** Let  $S: J \mapsto \mathbb{N}$  be a restricted pattern with  $\hat{j} = O(m/\ln m)$  and  $\sum_{j \in J} j^{-\alpha} (\ln j)^\beta = o(m^{1-\alpha} (\ln m)^{\beta-1})$ .

Assume  $r - d(k) = O(\ln m)$  and  $k = o(m(\ln m)^{\frac{1}{\alpha-1}})$ .

As  $m \rightarrow \infty$ , we have

$$P(X_n^{[r]}(S) > k) \sim P(S, n) \exp(-C(\rho; N_k \setminus J)) \sum_{j=0}^{r-1-d(k)} \frac{C^j(\rho; N_k \setminus J)}{j!}. \quad \square$$

## More results

**Corollary.** Let  $r = O(\ln n)$  and  $k = o\left(n(\ln n)^{\frac{1}{\alpha-1}}\right)$ . As  $n \rightarrow \infty$ , if there is no restricted pattern, we have

$$P(X_n^{[r]} > k) \sim \exp(-C(\rho; N_k)) \sum_{j=0}^{r-1} \frac{C^j(\rho; N_k)}{j!}.$$

**Theorem.** Let  $S: J \mapsto \mathbb{N}$  be a restricted pattern satisfying  $\hat{j} = o\left(m(\ln m)^{\frac{1}{\alpha-1}}\right)$  and  $\sum_{j \in J} \frac{(\ln j)^\beta}{j^\alpha} = o\left(m^{1-\alpha}(\ln m)^{\beta-1}\right)$ . When  $r = \sum_{j \in J} S(j) + 1$ , we have in the labeled case

$$E(X_n^{[r]}(S)) \sim P(S, n)m \exp(-c + C(\rho; J)),$$

and  $E(X_n^{[1]}) \sim ne^{-c}$ .

We have equivalent results for **unlabelled structures**.

# The case $\alpha = -p < 0$

In the following we consider the case

$$C(z) = d(1 - z/\rho)^{-p} + c + o(1)$$

as  $z \rightarrow \rho$  in the open disk  $|z| < \rho$ , where  $c, d$ , and  $p$  are constants with  $p, d > 0$ , and  $C(z)$  is analytic in the open disc  $|z| < \rho$ .

We define  $h(z) = d(1 - z)^{-p} + b \ln \frac{1}{1-z}$ . The asymptotics of  $[z^n] \exp(h(z))$  was studied by Wright (1949) and Hayman (1956).

**Condition:**  $\hat{j} = \max\{j : j \in J\}$ ,  $\hat{j} = o\left(m^{\frac{1}{p+1}}\right)$ ,  
 $\sum_{j \in J} j^{p-1} = o\left(m^{\frac{p}{6(p+1)}}\right)$ , and  $\sum_{j \in J} j^p = o\left(m^{\frac{1}{p+1}}\right)$ .

Finally, let  $R = R(m)$  be defined by

$$Rh'(R) = m, \quad 0 < R < 1.$$

# The case $\alpha = -p < 0$

We can extend the result of Wright and Hayman to generating functions including a restricted pattern  $S$ .

**Theorem.** Let  $\rho < 1$ ; suppose the pattern  $S$  satisfies the previous conditions, and let  $R$  be defined as before. Then as  $m \rightarrow \infty$  and for **unlabeled structures**, we have

$$[z^n]F(z; S) \sim \frac{1}{\sqrt{2\pi p(p+1)d}} \left( \prod_{j \in J} \binom{C_j + S(j) - 1}{S(j)} e^{-C_j \rho^j} \right) \\ (R\rho)^{-m} \left( \frac{m}{pd} \right)^{-\frac{p+2}{2(p+1)}} e^{C(R\rho) + r_0},$$

where  $r_0 = \sum_{k \geq 2} C(\rho^k)/k$ .  $\square$

**Corollary.** Suppose that in addition to the previous conditions the restricted pattern  $S = \left[ \prod_{j \in J} j^{S(j)} \right]$  also satisfies

$$\sum_{j \in J} j S(j) = o\left(n^{1/(p+1)}\right).$$

Then, as  $n \rightarrow \infty$ ,

$$\begin{aligned} P(S, n) &\sim \prod_{j \in J} P\left(\left[j^{S(j)}\right], n\right) \\ &\sim \prod_{j \in J} \binom{C_j + S(j) - 1}{S(j)} \rho^{j S(j)} (1 - \rho^j)^{C_j}. \quad \square \end{aligned}$$

We also have results for ***r*th smallest** sizes as well as for **labeled structures**.

# Examples

**Example 1.** A **rooted labeled tree** consists of a set of components (subtrees). The component generating function  $C(z)$  is of alg-log type at the singularity  $1/e$  with algebraic exponent  $\alpha = 1/2$ ,  $\beta = 0$ ,  $d = -\sqrt{2}$  and  $c = 1$ :

$$C(z) = 1 - \sqrt{2}(1 - ez)^{1/2} + O(1 - ez).$$

Consider the restricted pattern  $S = [2^s t^3]$ , where  $s = \lfloor \sqrt{n} \rfloor$ , and  $t = \lfloor \ln n \rfloor$ . We have

$$P(S, n) \sim \frac{1}{3!} \frac{(C_t e^{-t}/t!)^3}{\exp(C_t e^{-t}/t!)} \frac{1}{s!} \frac{(C_2 e^{-2}/2!)^s}{\exp(C_2 e^{-2}/2!)}.$$

Noting

$$C_2/2! = 1 \text{ and } C_t e^{-t}/t! \sim \frac{1}{\sqrt{2\pi}} t^{-3/2} \text{ as } t \rightarrow \infty,$$

we obtain

$$P(S, n) \sim \frac{1}{6}(2\pi)^{-3/2}(\ln n)^{-9/2} \frac{1}{s!} e^{-2s} \exp(-e^{-2}).$$

If in addition to the restricted pattern  $S$  we want information related to the **5th ( $r = 5$ ) smallest component size**, we get

$$P(X_n^{[5]}(S) > k) \sim P(S, n) \exp(-C(\rho; N_k \setminus J)) \sum_{j=0}^{4-d(k)} \frac{(C(\rho; N_k \setminus J))^j}{j!},$$

for  $k = o(n(\ln n)^{-2})$ . Since here  $d(k) = 0$  when  $k < 2$  and  $d(k) \geq s$  when  $k \geq 2$ , we have  $P(X_n^{[5]}(S) > k) = 0$  when  $k \geq 2$ , and

$$P\left(X_n^{[5]}(S) > 1\right) \sim P(S, n) \exp(-1/e) \sum_{j=0}^4 \frac{e^{-j}}{j!}.$$

The **expected size of the smallest subtree** is asymptotic to  $n/e$ .

**Example 2.** *Fragmented permutation* are the combinatorial class  $\mathcal{F} = \text{SET}(\text{SEQ}_{\geq 1}(\mathcal{L}))$ . They correspond to unordered collections of permutations. The generating function of fragmented permutations is  $F(z) = \exp\left(\frac{z}{1-z}\right)$ . The coefficients form the sequence A000262 in Sloane's Encyclopedia of Integer Sequences that counts sets of lists.

The component generating function is  $C(z) = \frac{z}{1-z} = \frac{1}{1-z} - 1$  and so it is of alg-log type with  $\alpha = -1$ ,  $\rho = 1$ ,  $d = 1$  and  $c = -1$ . We have

$$[z^n]F(z) = \frac{F_n}{n!} \sim \frac{e^{-1/2}e^{2\sqrt{n}}}{2\sqrt{\pi}n^{3/4}},$$

and the saddle point is

$$R = 1 - \frac{1}{\sqrt{n}} + \frac{1}{2n} + O(n^{-3/2}).$$

Now consider fragmented permutations with the restricted pattern  $S = [2^s t^3]$ , where  $s = \lfloor n^{1/4} \rfloor$  and  $t = \lfloor \ln n \rfloor$ . The probability that a random fragmented permutation has this restricted pattern  $S$  is

$$P(S, n) \sim \frac{1}{3!} (C_t/t!)^3 \exp(-C_t/t!) \frac{1}{s!} (C_2/2!)^s \exp(-C_2/2!).$$

We note that  $C_j = j!$ , and we obtain

$$P(S, n) \sim \frac{1}{6} \frac{1}{s!} e^{-2}.$$

We have  $C(\rho, N_k) = k$ , and the **expected size of the smallest component** is asymptotic to  $\sum_{k \geq 0} e^{-k} = \frac{e}{e-1}$ .

# Summary

Let  $Y_n$  denote the number of components in a random structure of size  $n$ , and  $X_n$  be the smallest component size.

- When  $C(z)$  is of logarithmic type,  $Y_n$  is asymptotically normal with expected value and variance both proportional to  $\ln n$ .  $E(X_n)$  is also proportional to  $\ln n$ .
- When  $C(z)$  is of alg-log type with algebraic exponent  $0 < \alpha < 1$ , we have  $P(Y_n = k) \sim \frac{e^{-c} c^{k-1}}{(k-1)!}$ . That is,  $Y_n - 1$  is asymptotically Poisson with mean  $c$ .  $E(X_n)$  is asymptotic to  $ne^{-c}$ .
- When  $C(z)$  is of alg-log type with algebraic exponent  $\alpha = -p < 0$ ,  $Y_n$  is also asymptotically normal with mean and variance both proportional to  $n^{p/(p+1)}$ .  $E(X_n)$  is asymptotic to a constant.