Analysis of label-dependent parameters in increasing trees and generalizations

Alois Panholzer

Institute of Discrete Mathematics and Geometry
Vienna University of Technology
Alois.Panholzer@tuwien.ac.at

CanaDAM 2009, Montreal, 25.-28. 5. 2009
Outline of the talk

1. Simple families of increasing trees
2. Results
3. Proof
4. Extensions and generalizations
Simple families of increasing trees
Simple increasing tree families:

- Class of **labelled** trees

- Used as **model** in various fields of **application**:
  - Pyramid schemes
  - Spread of epidemics
  - Stemma construction of philology
  - Physical particle aggregation and disaggregation
  - Stochastic growth of networks (e.g., www)

- Applications in **computer science**:
  - Basis of Burge’s sorting method
  - Data structure for mergeable priority queues
Simple increasing tree families:

- Class of labelled trees
- Used as model in various fields of application:
  - Pyramid schemes
  - Spread of epidemics
  - Stemma construction of philology
  - Physical particle aggregation and disaggregation
  - Stochastic growth of networks (e.g., www)

Applications in computer science:
- Basis of Burge’s sorting method
- Data structure for mergeable priority queues
Simple families of increasing trees: Applications

Simple increasing tree families:

- Class of labelled trees
- Used as model in various fields of application:
  - Pyramid schemes
  - Spread of epidemics
  - Stemma construction of philology
  - Physical particle aggregation and disaggregation
  - Stochastic growth of networks (e.g., www)
- Applications in computer science:
  - Basis of Burge’s sorting method
  - Data structure for mergeable priority queues
Simple families of increasing trees: Definition

Simple increasing tree families:

- Labelled trees: size-$n$ tree labelled with $1, 2, \ldots, n$
- Increasingly labelled: labels from root to arbitrary node are increasing
- Underlying unlabelled tree model: simply generated trees ($\simeq$ Galton-Watson trees)

Contain several important tree families:

- Recursive trees
- Plane-oriented recursive trees (PORTs)
- Binary increasing trees ($\simeq$ binary search trees)
Simple increasing tree families:

- **Labelled trees**: size-\(n\) tree labelled with 1, 2, \ldots, \(n\)
- Increasingly labelled: labels from root to arbitrary node are increasing
- Underlying unlabelled tree model: simply generated trees (\(\approx\) Galton-Watson trees)

Contain several important tree families:

- Recursive trees
- Plane-oriented recursive trees (PORTs)
- Binary increasing trees (\(\approx\) binary search trees)
Simple increasing tree families:

- **Labelled trees**: size-$n$ tree labelled with $1, 2, \ldots, n$
- **Increasingly labelled**: labels from root to arbitrary node are increasing

Underlying unlabelled tree model:

- simply generated trees (≈ Galton-Watson trees)

Contain several important tree families:

- Recursive trees
- Plane-oriented recursive trees (PORTs)
- Binary increasing trees (≈ binary search trees)
Simple increasing tree families:

- **Labelled trees**: size-$n$ tree labelled with $1, 2, \ldots, n$
- **Increasingly labelled**: labels from root to arbitrary node are increasing
- **Underlying unlabelled tree model**: simply generated trees (\(\simeq\) Galton-Watson trees)

Contain several important tree families:

- Recursive trees
- Plane-oriented recursive trees (PORTs)
- Binary increasing trees (\(\simeq\) binary search trees)
Simple increasing tree families:

- **Labelled trees:** size-\( n \) tree labelled with 1, 2, \ldots, \( n \)
- **Increasingly labelled:** labels from root to arbitrary node are increasing
- **Underlying unlabelled tree model:**
  - *simply generated trees* (\( \simeq \) Galton-Watson trees)

**Contain several important tree families:**

- **Recursive** trees
- **Plane-oriented recursive** trees (PORTs)
- **Binary** increasing trees (\( \simeq \) binary search trees)
Simple families of increasing trees: Examples

Recursive trees

- **Non-plane** increasingly labelled rooted trees

- Number of different recursive trees: $T_n = (n - 1)!$

- Exponential generating function: $T(z) = \log \frac{1}{1-z}$

All 6 recursive trees of size 4.
**Recursive trees**

- **Non-plane** increasingly labelled rooted trees
- **Number** of different recursive trees: $T_n = (n - 1)!$
- Exponential generating function: $T(z) = \log \frac{1}{1-z}$

All 6 recursive trees of size 4.
Recursive trees

- **Non-plane** increasingly labelled rooted trees
- **Number** of different recursive trees: $T_n = (n - 1)!$
- Exponential generating function: $T(z) = \log \frac{1}{1-z}$

All 6 recursive trees of size 4.
Simple families of increasing trees: Examples

**Simple tree evolution process** for generating random recursive trees:

- Start with root labelled by 1
- **Step j**: node with label $j$ is attached to any previous node with equal probability $1/(j - 1)$

After $n$ steps $\Rightarrow$

every size-$n$ recursive tree has equal probability $\frac{1}{(n-1)!}$
Simple families of increasing trees: Examples

**Simple tree evolution process** for generating *random recursive trees*:

- Start with root labelled by 1
- Step $j$: node with label $j$ is attached to any previous node with equal probability $1/(j - 1)$

After $n$ steps $\Rightarrow$

every size-$n$ recursive tree has equal probability $\frac{1}{(n - 1)!}$
Simple tree evolution process for generating random recursive trees:

- Start with root labelled by 1
- Step $j$: node with label $j$ is attached to any previous node with equal probability $1/(j - 1)$

After $n$ steps $\Rightarrow$

every size-$n$ recursive tree has equal probability $\frac{1}{(n - 1)!}$
**Simple tree evolution process** for generating random recursive trees:

- Start with root labelled by 1
- **Step j:** node with label $j$ is attached to any previous node with equal probability $1/(j - 1)$

**After $n$ steps** $\Rightarrow$

Every size-$n$ recursive tree has equal probability $\frac{1}{(n - 1)!}$
Simple families of increasing trees: Examples

Random generation of recursive trees of size 4 by tree evolution process:
Simple families of increasing trees: Examples

Random generation of recursive trees of size 4 by tree evolution process:
Simple families of increasing trees: Examples

Random generation of recursive trees of size 4 by tree evolution process:

```
1
   \rightarrow p = 1
   \downarrow
1
   \rightarrow 2
   \rightarrow
```
Simple families of increasing trees: Examples

Random generation of recursive trees of size 4 by tree evolution process:
Simple families of increasing trees: Examples

Random generation of recursive trees of size 4 by tree evolution process:
**Simple families of increasing trees: Examples**

**Plane-oriented recursive trees (PORTs)**
- **Plane** increasingly labelled rooted trees
- Number of trees: $T_n = 1 \cdot 3 \cdot 5 \cdots (2n - 3) = (2n - 3)!!$
- Exponential generating function: $T(z) = 1 - \sqrt{1 - 2z}$

All 15 plane-oriented recursive trees of size 4.
Simple families of increasing trees: Examples

Plane-oriented recursive trees (PORTs)
- **Plane** increasingly labelled rooted trees
- **Number of trees:** \( T_n = 1 \cdot 3 \cdot 5 \cdots (2n - 3) = (2n - 3)!! \)
- **Exponential generating function:** \( T(z) = 1 - \sqrt{1 - 2z} \)

All 15 plane-oriented recursive trees of size 4.
Simple families of increasing trees: Examples

Plane-oriented recursive trees (PORTs)

- Plane increasingly labelled rooted trees
- Number of trees: \( T_n = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 3) = (2n - 3)!! \)
- Exponential generating function: \( T(z) = 1 - \sqrt{1 - 2z} \)

All 15 plane-oriented recursive trees of size 4.
Simple families of increasing trees: Examples

**Simple tree evolution process** for generating random plane-oriented recursive trees:

- Start with root labelled by 1
- **Step $j$:** node with label $j$ is attached to any previous node of out-degree $d$ with probability $\left(\frac{d + 1}{2j - 3}\right)$

After $n$ steps $\Rightarrow$

every size-$n$ recursive tree has equal probability $\frac{1}{(2n - 3)!!}$
Simple tree evolution process for generating random plane-oriented recursive trees:

- Start with root labelled by 1
- Step $j$: node with label $j$ is attached to any previous node of out-degree $d$ with probability $(d + 1)/(2j - 3)$

After $n$ steps $\Rightarrow$

every size-$n$ recursive tree has equal probability $\frac{1}{(2n - 3)!!}$
Simple tree evolution process for generating random plane-oriented recursive trees:

- Start with root labelled by 1
- Step $j$: node with label $j$ is attached to any previous node of out-degree $d$ with probability $(d + 1)/(2j - 3)$

After $n$ steps $\Rightarrow$

every size-$n$ recursive tree has equal probability $\frac{1}{(2n - 3)!!}$
Simple families of increasing trees: Examples

**Simple tree evolution process** for generating random plane-oriented recursive trees:

- Start with root labelled by 1
- **Step** $j$: node with label $j$ is attached to any previous node of out-degree $d$ with probability $(d + 1)/(2j - 3)$

After $n$ steps $\Rightarrow$

every size-$n$ recursive tree has equal probability $\frac{1}{(2n - 3)!!}$
Simple families of increasing trees: Examples

Random generation of plane-oriented recursive trees by tree evolution process:
Simple families of increasing trees: Examples

Random generation of plane-oriented recursive trees by tree evolution process:
Simple families of increasing trees: Examples

Random generation of plane-oriented recursive trees by tree evolution process:

```
     1
  __________
     |      |
     2     p = 1
```

1

2

p = 1
Simple families of increasing trees: Examples

Random generation of plane-oriented recursive trees by tree evolution process:
Simple families of increasing trees: Examples

Random generation of plane-oriented recursive trees by tree evolution process:
Simple families of increasing trees: Examples

Binary increasing trees $T_n = n!$

- Binary trees equipped with increasing labellings
- Number of trees: $T_n = n!$

Unlabelled binary trees of size 3

\[\downarrow \text{ increasing labellings}\]
Simple families of increasing trees: Examples

**Binary increasing trees** $T_n = n!$

- **Binary trees** equipped with increasing labellings
- **Number of trees**: $T_n = n!$

Unlabelled binary trees of size 3

↓ increasing labellings
Simple families of increasing trees: Examples

**Binary increasing trees** $T_n = n!$
- Binary trees equipped with increasing labellings
- Number of trees: $T_n = n!$

Unlabelled binary trees of size 3

![Diagram of unlabelled binary trees of size 3 with increasing labellings](image)
Simple families of increasing trees: Examples

Binary increasing trees $T_n = n!$
- Binary trees equipped with increasing labellings
- Number of trees: $T_n = n!$

Unlabelled binary trees of size 3
↓ increasing labellings
**General simple increasing trees** [Bergeron et al, 1992]

**Class** $\mathcal{T}$ of simple increasing trees:

- **Degree weight sequence:** $(\varphi_i)_{i \geq 0}$
- **Weight of node according out-degree $d(v)$:**

$$\varphi d(v)$$

- **Weight of tree:** product of node-weights

$$w(T) = \prod_{v \in T} \varphi d(v)$$

- **Number of increasing labellings**

$$L(T)$$
**Simple families of increasing trees: General setting**

**General simple increasing trees** [Bergeron et al, 1992]

**Class** $\mathcal{T}$ of simple increasing trees:

- **Degree weight sequence**: $(\varphi_i)_{i \geq 0}$
- **Weight of node according out-degree $d(v)$**:
  $$\varphi d(v)$$

- **Weight of tree**: product of node-weights
  $$w(T) = \prod_{v \in T} \varphi d(v)$$

- **Number of increasing labellings**
  $$L(T)$$
General simple increasing trees [Bergeron et al, 1992]

Class $\mathcal{T}$ of simple increasing trees:

- Degree weight sequence: $(\varphi_i)_{i \geq 0}$
- Weight of node according out-degree $d(v)$:

$$
\varphi d(v)
$$

- Weight of tree: product of node-weights

$$
\mathcal{W}(T) = \prod_{v \in T} \varphi d(v)
$$

- Number of increasing labellings

$$
\mathcal{L}(T)
$$
Simple families of increasing trees: General setting

**General simple increasing trees** [Bergeron et al, 1992]

**Class** $\mathcal{T}$ of simple increasing trees:

- Degree weight sequence: $(\varphi_i)_{i \geq 0}$
- Weight of node according out-degree $d(v)$:

$$\varphi d(v)$$

- Weight of tree: product of node-weights

$$w(T) = \prod_{v \in T} \varphi d(v)$$

- Number of increasing labellings

$$L(T)$$
Simple families of increasing trees: General setting

**General simple increasing trees** [Bergeron et al, 1992]

**Class** $\mathcal{T}$ of simple increasing trees:

- Degree weight sequence: $(\varphi_i)_{i \geq 0}$
- Weight of node according out-degree $d(v)$:

\[
\varphi d(v)
\]

- Weight of tree: product of node-weights

\[
w(T) = \prod_{v \in T} \varphi d(v)
\]

- Number of increasing labellings

\[
L(T)
\]
Simple families of increasing trees: General setting

General simple increasing trees [Bergeron et al, 1992]

Class $\mathcal{T}$ of simple increasing trees:
- Degree weight sequence: $(\varphi_i)_{i \geq 0}$
- Weight of node according out-degree $d(v)$:

$$\varphi d(v)$$

- Weight of tree: product of node-weights

$$w(T) = \prod_{v \in T} \varphi d(v)$$

- Number of increasing labellings

$$L(T)$$
Simple families of increasing trees: General setting

- **Probability model:** Choose tree proportional to $w(T) \cdot L(T)$

- **Total weight:** $T_n = \sum_{|T|=n} w(T) \cdot L(T)$

- **Generating functions setup:**
  - Degree weight GF: $\varphi(t) = \sum_{i \geq 0} \varphi_i t^i$
    - Binary increasing trees: $\varphi(t) = (1 + t)^2$
    - Recursive trees: $\varphi(t) = \exp(t)$
    - Plane-oriented recursive trees: $\varphi(t) = 1/(1 - t)$
  - Exponential GF: $T(z) = \sum_{n \geq 1} T_n z^n / n!$

Satisfies differential equation $T'(z) = \varphi(T(z))$
Simple families of increasing trees: General setting

- **Probability model:** Choose tree proportional to $w(T) \cdot L(T)$

- **Total weight:** $T_n = \sum_{|T|=n} w(T) \cdot L(T)$

- **Generating functions setup:**
  - Degree weight GF: $\phi(t) = \sum_{i \geq 0} \phi_i t^i$
    - Binary increasing trees: $\phi(t) = (1 + t)^2$
    - Recursive trees: $\phi(t) = \exp(t)$
    - Plane-oriented recursive trees: $\phi(t) = 1/(1 - t)$
  - Exponential GF: $T(z) = \sum_{n \geq 1} T_n z^n / n!$
    - Satisfies differential equation $T'(z) = \phi(T(z))$
Simple families of increasing trees: General setting

- Probability model: Choose tree proportional to $w(T) \cdot L(T)$
- Total weight: $T_n = \sum_{|T|=n} w(T) \cdot L(T)$
- Generating functions setup:
  - Degree weight GF: $\varphi(t) = \sum_{i \geq 0} \varphi_i t^i$
    - Binary increasing trees: $\varphi(t) = (1 + t)^2$
    - Recursive trees: $\varphi(t) = \exp(t)$
    - Plane-oriented recursive trees: $\varphi(t) = 1/(1 - t)$
  - Exponential GF: $T(z) = \sum_{n \geq 1} T_n z^n / n!$

Satisfies differential equation $T'(z) = \varphi(T(z))$
Simple families of increasing trees: General setting

- **Probability model:** Choose tree proportional to $w(T) \cdot L(T)$
- **Total weight:** $T_n = \sum_{|T|=n} w(T) \cdot L(T)$
- **Generating functions setup:**
  - **Degree weight GF:** $\varphi(t) = \sum_{i \geq 0} \varphi_i t^i$
    - **Binary increasing trees:** $\varphi(t) = (1 + t)^2$
    - **Recursive trees:** $\varphi(t) = \exp(t)$
    - **Plane-oriented recursive trees:** $\varphi(t) = 1/(1 - t)$
  - **Exponential GF:** $T(z) = \sum_{n \geq 1} T_n z^n / n!$
  - Satisfies differential equation $T'(z) = \varphi(T(z))$
Simple families of increasing trees: General setting

- **Probability model**: Choose tree proportional to \( w(T) \cdot L(T) \)
- **Total weight**: \( T_n = \sum_{|T|=n} w(T) \cdot L(T) \)
- **Generating functions setup**:
  - **Degree weight GF**: \( \varphi(t) = \sum_{i \geq 0} \varphi_i t^i \)
    - **Binary increasing trees**: \( \varphi(t) = (1 + t)^2 \)
    - **Recursive trees**: \( \varphi(t) = \exp(t) \)
    - **Plane-oriented recursive trees**: \( \varphi(t) = 1/(1 - t) \)
  - **Exponential GF**: \( T(z) = \sum_{n \geq 1} T_n z^n / n! \)

Satisfies differential equation

\[
T'(z) = \varphi(T(z))
\]
Results
Results: Characterization of evolving trees

Question: Which simple increasing tree families are generated by tree evolution process?

Theorem (Panholzer & Prodinger, 2007)

Simple increasing tree family generated via tree evolution process

iff
degree-weight generating function \( \tilde{\varphi}(t) \) given by:

\[
\tilde{\varphi}(t) = a\varphi(bt), \quad a, b > 0, \quad \text{with } \varphi(t) \text{ as follows:}
\]

- Recursive trees: \( \varphi(t) = e^t \),
- \( d \)-ary trees: \( \varphi(t) = (1 + t)^d, \quad d \in \{2, 3, 4, \ldots \} \),
- Generalized PORTs: \( \varphi(t) = \frac{1}{(1 - t)^\alpha}, \quad \alpha > 0 \).
Simple families of increasing trees

Results

Question: Which simple increasing tree families are generated by tree evolution process?

Theorem (Panholzer & Prodinger, 2007)

Simple increasing tree family generated via tree evolution process

iff

degree-weight generating function \( \tilde{\varphi}(t) \) given by:

\[ \tilde{\varphi}(t) = a\varphi(bt), \quad a, b > 0, \quad \text{with } \varphi(t) \text{ as follows:} \]

- Recursive trees: \( \varphi(t) = e^t \),
- \( d \)-ary trees: \( \varphi(t) = (1 + t)^d, \quad d \in \{2, 3, 4, \ldots\} \),
- Generalized PORTs: \( \varphi(t) = \frac{1}{(1 - t)^\alpha}, \quad \alpha > 0. \)
**Results: Characterization of evolving trees**

**Tree evolution process** for generating random evolving increasing trees:

- Start with root labelled by 1
- **Step** $j$: node with label $j$ is attached to any previous node of out-degree $d$ with probability proportional to $\frac{(d + 1)\varphi_{d+1}}{\varphi_d}$

After $n$ steps $\Rightarrow$ every size-$n$ increasing tree has equal probability

$$\frac{1}{\kappa^{n-1}(n-1)!\binom{n-1+c}{n-1}}, \quad \kappa, c \text{ dependent on } \varphi(t)$$
**Results: Characterization of evolving trees**

**Tree evolution process** for generating random evolving increasing trees:

- **Start with** root labelled by 1
- **Step** $j$: node with label $j$ is attached to any previous node of out-degree $d$ with probability proportional to

$$\frac{(d + 1)\varphi_{d+1}}{\varphi_d}$$

After $n$ steps $\Rightarrow$ every size-$n$ increasing tree has equal probability

$$\frac{1}{\kappa^{n-1}(n-1)!(\frac{n-1+c}{n-1})}, \quad \kappa, c \text{ dependent on } \varphi(t)$$
**Results: Characterization of evolving trees**

**Tree evolution process** for generating random evolving increasing trees:

- **Start with** root labelled by 1
- **Step j:** node with label $j$ is attached to any previous node of out-degree $d$ with probability proportional to

  \[
  \frac{(d + 1)\varphi_{d+1}}{\varphi_d}
  \]

After $n$ steps $\Rightarrow$ every size-$n$ increasing tree has equal probability

\[
\frac{1}{\kappa^{n-1}(n-1)!\left(\frac{n-1+c}{n-1}\right)}, \quad \kappa, c \text{ dependent on } \varphi(t)
\]
**Results: Characterization of evolving trees**

**Tree evolution process** for generating random evolving increasing trees:

- Start with **root labelled by 1**
- **Step $j$:** node with label $j$ is attached to any previous node of out-degree $d$ with probability proportional to

$$\frac{(d + 1)\varphi_{d+1}}{\varphi_d}$$

**After $n$ steps** $\Rightarrow$ every size-$n$ increasing tree has equal probability

$$\frac{1}{\kappa^{n-1}(n-1)!\binom{n-1+c}{n-1}}, \quad \kappa, c \text{ dependent on } \varphi(t)$$
Results: Label-dependent parameters

Label-dependent parameters for random size-$n$ increasing trees:

- Depth (level; # ancestors) of node $j$
- Distance between nodes $j_1$ and $j_2$
- Subtree-size (# descendants) of node $j$
- Out-degree of node $j$
- Number of leaves in subtree rooted at node $j$

Interesting for evolving trees:

Behaviour of $j$-th individual in tree evolution process
Results: Label-dependent parameters

Label-dependent parameters for random size-$n$ increasing trees:

- **Depth** (level; \# ancestors) of node $j$
- **Distance** between nodes $j_1$ and $j_2$
- **Subtree-size** (\# descendants) of node $j$
- **Out-degree** of node $j$
- **Number of leaves** in subtree rooted at node $j$

Interesting for evolving trees:

Behaviour of $j$-th individual in tree evolution process
Results: Label-dependent parameters

- Depth of node 5: \( D(5) = 1 \)
- Distance between nodes 4 and 5: \( \Delta(4, 5) = 3 \)
- Subtree-size of node 5: \( S(5) = 6 \)
- Out-degree of node 5: \( R(5) = 3 \)
- Number of Leaves in subtree rooted at 5: \( L(5) = 4 \)
Results: Label-dependent parameters

- **Depth** of node 5: $D(5) = 1$
- Distance between nodes 4 and 5: $\Delta(4, 5) = 3$
- Subtree-size of node 5: $S(5) = 6$
- Out-degree of node 5: $R(5) = 3$
- Number of Leaves in subtree rooted at 5: $L(5) = 4$
**Results: Label-dependent parameters**

- **Depth** of node 5: $D(5) = 1$
- **Distance** between nodes 4 and 5: $\Delta(4, 5) = 3$
- **Subtree-size** of node 5: $S(5) = 6$
- **Out-degree** of node 5: $R(5) = 3$
- **Number of Leaves** in subtree rooted at 5: $L(5) = 4$
Results: Label-dependent parameters

- **Depth** of node 5: $D(5) = 1$
- **Distance** between nodes 4 and 5: $\Delta(4, 5) = 3$
- **Subtree-size** of node 5: $S(5) = 6$
- **Out-degree** of node 5: $R(5) = 3$
- **Number of Leaves** in subtree rooted at 5: $L(5) = 4$
Results: Label-dependent parameters

- Depth of node 5: $D(5) = 1$
- Distance between nodes 4 and 5: $\Delta(4, 5) = 3$
- Subtree-size of node 5: $S(5) = 6$
- Out-degree of node 5: $R(5) = 3$
- Number of Leaves in subtree rooted at 5: $L(5) = 4$
**Results:** Label-dependent parameters

- **Depth** of node 5: $D(5) = 1$
- **Distance** between nodes 4 and 5: $\Delta(4, 5) = 3$
- **Subtree-size** of node 5: $S(5) = 6$
- **Out-degree** of node 5: $R(5) = 3$
- **Number of Leaves** in subtree rooted at 5: $L(5) = 4$
**Results: Depth**

**Theorem (Panholzer & Prodinger, 2007)**

Depth $D_{n,j}$ of node $j$ for evolving increasing tree satisfies:

*Explicit formula for probabilities* $\mathbb{P}\{D_{n,j} = m\}$:

$$\mathbb{P}\{D_{n,j} = m\} = \frac{(c+1)^m}{(c+1)^{j-1}} \binom{j-1}{m}.$$  

Normalized random variable *asymp. Gaussian distributed*:

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left\{ \frac{D_{n,j} - (c+1) \log j}{\sqrt{(c+1) \log j}} \leq x \right\} - \Phi(x) \right| = O\left(\frac{1}{\sqrt{\log j}}\right).$$
**Results: Depth**

**Theorem (Panholzer & Prodinger, 2007)**

Depth $D_{n,j}$ of node $j$ for evolving increasing tree satisfies:

*Explicit formula for probabilities* $\mathbb{P}\{D_{n,j} = m\}$:

$$\mathbb{P}\{D_{n,j} = m\} = \frac{(c + 1)^m}{(c + 1)^{j-1} \binom{j-1}{m}}.$$

*Normalized random variable asymp. Gaussian distributed:*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left\{ \frac{D_{n,j} - (c + 1) \log j}{\sqrt{(c + 1) \log j}} \leq x \right\} - \Phi(x) \right| = \mathcal{O}\left( \frac{1}{\sqrt{\log j}} \right).$$
**Results:** Distance

**Theorem (Kuba & Panholzer, 2008+)**

Distance $\Delta_{n:j_1,j_2}$ between nodes $j_1$ and $j_2$ satisfies:

**Normalized random variable asymp. Gaussian distributed:**

\[
\frac{\Delta_{n:j_1,j_2} - (c + 1)(\log j_1 + \log j_2)}{\sqrt{(c + 1)(\log j_1 + \log j_2)}} \xrightarrow{(d)} \mathcal{N}(0, 1).
\]

Explicit formulæ for $\mathbb{E}$ and $\mathbb{V}$ (e.g., PORT):

\[
\mathbb{E}(\Delta_{n:j_1,j_2}) = H_{2j_2-2} + H_{2j_1} - \frac{1}{2} H_{j_2-1} - \frac{1}{2} H_{j_1} - 1,
\]

\[
\mathbb{V}(\Delta_{n:j_1,j_2}) = H_{2j_2-2} + H_{2j_1} - \frac{1}{2} H_{j_2-1} - \frac{1}{2} H_{j_1} - H^{(2)}_{2j_2-2} - 3H^{(2)}_{2j_1} + \frac{1}{4} H^{(2)}_{j_2-1} + \frac{3}{4} H^{(2)}_{j_1-1} + 2.
\]
**Results:** Distance

**Theorem (Kuba & Panholzer, 2008+)**

Distance $\Delta_{n; j_1, j_2}$ between nodes $j_1$ and $j_2$ satisfies:

**Normalized random variable asymp. Gaussian distributed:**

$$
\frac{\Delta_{n; j_1, j_2} - (c + 1)(\log j_1 + \log j_2)}{\sqrt{(c + 1)(\log j_1 + \log j_2)}} \overset{(d)}{\to} \mathcal{N}(0, 1).
$$

**Explicit formulæ for $\mathbb{E}$ and $\mathbb{V}$ (e.g., PORT):**

$$
\mathbb{E}(\Delta_{n; j_1, j_2}) = H_{2j_2 - 2} + H_{2j_1} - \frac{1}{2} H_{j_2 - 1} - \frac{1}{2} H_{j_1} - 1,
$$

$$
\mathbb{V}(\Delta_{n; j_1, j_2}) = H_{2j_2 - 2} + H_{2j_1} - \frac{1}{2} H_{j_2 - 1} - \frac{1}{2} H_{j_1}
- H_{2j_2 - 2}^{(2)} - 3H_{2j_1}^{(2)} + \frac{1}{4} H_{j_2 - 1}^{(2)} + \frac{3}{4} H_{j_1 - 1}^{(2)} + 2.
$$
**Results: Subtree-size**

**Theorem (Kuba & Panholzer, 2006)**

Size $S_{n,j}$ of subtree rooted at $j$ satisfies:

*Explicit formula for probabilities* $\mathbb{P}\{S_{n,j} = m\}$:

$$\mathbb{P}\{S_{n,j} = m\} = \frac{(j-1+c) (m-1+c) (n-m-1)}{(j-1) (m-1+c) (n-1) (n-1+c)}.$$  

Limiting distribution results (4 regions of $j$):

- **$j$ fixed**: $\frac{S_{n,j}}{n}$ asympt. Beta-distributed
  $$\frac{S_{n,j}}{n} \overset{(d)}{\to} \beta(c + 1, j - 1)$$
Theorem (Kuba & Panholzer, 2006)

Size $S_{n,j}$ of subtree rooted at $j$ satisfies:

*Explicit formula for probabilities* $P\{S_{n,j} = m\}$:

$$P\{S_{n,j} = m\} = \frac{(j-1+c)(m-1+c)(n-m-1)}{(n-1)(n-1+c)(j-1)(n-m-1)}.$$ 

*Limiting distribution results (4 regions of $j$):

- $j$ fixed: $\frac{S_{n,j}}{n}$ asymp. Beta-distributed

$$\frac{S_{n,j}}{n} \xrightarrow{(d)} \beta(c + 1, j - 1)$$
Results: Subtree-size

Theorem (Kuba & Panholzer, 2006)

- **j small:** $j \to \infty$ such that $j = o(n)$:

  \[ \frac{j}{n} S_{n,j} \text{ asymp. } \text{Gamma-distributed} \]

  \[ \frac{j}{n} S_{n,j} \xrightarrow{(d)} \gamma(c + 1, 1) \]

- **central region:** $j \sim \rho n$, with $0 < \rho < 1$:

  $S_{n,j} \text{ asymp. negative binomial-distributed}$

  \[ S_{n,j} - 1 \xrightarrow{(d)} \text{NegBin}(c + 1, \rho) \]

- **j large:** $n - j = o(n)$: asymptotically $j$ has no children

  \[ S_{n,j} \xrightarrow{(d)} X, \text{ with } \mathbb{P}\{X = 1\} = 1 \]
**Results: Subtree-size**

**Theorem (Kuba & Panholzer, 2006)**

- **j small**: $j \rightarrow \infty$ such that $j = o(n)$:
  \[
  \frac{j}{n} S_{n,j} \xrightarrow{\text{asymp.}} \text{Gamma-distributed}
  \]
  \[
  \frac{j}{n} S_{n,j} \xrightarrow{(d)} \gamma(c + 1, 1)
  \]

- **central region**: $j \sim \rho n$, with $0 < \rho < 1$:
  \[
  S_{n,j} \xrightarrow{\text{asymp.}} \text{negative binomial-distributed}
  \]
  \[
  S_{n,j} - 1 \xrightarrow{(d)} \text{NegBin}(c + 1, \rho)
  \]

- **j large**: $n - j = o(n)$: asymptotically $j$ has no children
  \[
  S_{n,j} \xrightarrow{(d)} X, \text{ with } \mathbb{P}\{X = 1\} = 1
  \]
Results: Subtree-size

Theorem (Kuba & Panholzer, 2006)

- **j small**: \( j \to \infty \) such that \( j = o(n) \):
  \[
  \frac{j}{n} S_{n,j} \quad \text{asymp. Gamma-distributed}
  \]
  \[
  \frac{j}{n} S_{n,j} \xrightarrow{(d)} \gamma(c + 1, 1)
  \]

- **central region**: \( j \sim \rho n \), with \( 0 < \rho < 1 \):
  \( S_{n,j} \quad \text{asymp. negative binomial-distributed} \)
  \[
  S_{n,j} - 1 \xrightarrow{(d)} \text{NegBin}(c + 1, \rho)
  \]

- **j large**: \( n - j = o(n) \): asymptotically \( j \) has no children
  \[
  S_{n,j} \xrightarrow{(d)} X, \quad \text{with } \mathbb{P}\{X = 1\} = 1
  \]
Results: Out-degree

**Theorem (Kuba & Panholzer, 2007)**

Out-degree $R_{n,j}$ of node $j$ satisfies for recursive trees:

- **j small:** $j = o(n)$: asympt. **Gaussian distributed**

  \[
  \frac{R_{n,j} - (\log n - \log j)}{\sqrt{\log n - \log j}} \xrightarrow{(d)} \mathcal{N}(0, 1)
  \]

- **central region:** $j \sim \rho n$, with $0 < \rho < 1$:
  
  $R_{n,j}$ asymp. **Poisson-distributed**

  \[
  R_{n,j} \xrightarrow{(d)} \text{Poisson}(- \log \rho)
  \]

- **j large:** $n - j = o(n)$:

  \[
  R_{n,j} \xrightarrow{(d)} X, \text{ with } \mathbb{P}\{X = 0\} = 1
  \]
Results: Out-degree

Theorem (Kuba & Panholzer, 2007)

Out-degree $R_{n,j}$ of node $j$ satisfies for recursive trees:

- **$j$ small:** $j = o(n)$: asympt. *Gaussian distributed*

  $$\frac{R_{n,j} - (\log n - \log j)}{\sqrt{\log n - \log j}} \xrightarrow{(d)} \mathcal{N}(0, 1)$$

- **Central region:** $j \sim \rho n$, with $0 < \rho < 1$:

  $R_{n,j}$ asympt. *Poisson-distributed*

  $$R_{n,j} \xrightarrow{(d)} \text{Poisson}(-\log \rho)$$

- **$j$ large:** $n - j = o(n)$:

  $$R_{n,j} \xrightarrow{(d)} X, \text{ with } \mathbb{P}\{X = 0\} = 1$$
Results: Out-degree

Theorem (Kuba & Panholzer, 2007)

Out-degree $R_{n,j}$ of node $j$ satisfies for recursive trees:

- **j small:** $j = o(n)$: asympt. Gaussian distributed

\[
\frac{R_{n,j} - (\log n - \log j)}{\sqrt{\log n - \log j}} \xrightarrow{d} \mathcal{N}(0, 1)
\]

- **central region:** $j \sim \rho n$, with $0 < \rho < 1$:

$R_{n,j}$ asymp. Poisson-distributed

\[
R_{n,j} \xrightarrow{(d)} \text{Poisson}(\log \rho)
\]

- **j large:** $n - j = o(n)$:

\[
R_{n,j} \xrightarrow{(d)} X, \text{ with } \mathbb{P}\{X = 0\} = 1
\]
Results: Out-degree

Theorem (Kuba & Panholzer, 2007)

Out-degree $R_{n,j}$ of node $j$ satisfies for $d$-ary trees:

- **$j$ small:** $j = o(n)$: asymp. $j$ has **maximal out-degree** $d$

  \[ R_{n,j} \xrightarrow{(d)} X, \text{ with } P\{X = d\} = 1 \]

- **Central region:** $j \sim \rho n$, with $0 < \rho < 1$:

  $R_{n,j}$ asymp. **binomial-distributed**

  \[ R_{n,j} \xrightarrow{(d)} \text{Binomial}(d, 1 - \rho^{\frac{1}{d-1}}) \]

- **$j$ large:** $n - j = o(n)$:

  \[ R_{n,j} \xrightarrow{(d)} X, \text{ with } P\{X = 0\} = 1 \]
Results: Out-degree

Theorem (Kuba & Panholzer, 2007)

Out-degree $R_{n,j}$ of node $j$ satisfies for $d$-ary trees:

- **j small:** $j = o(n)$: asymp. $j$ has **maximal out-degree** $d$
  
  $$R_{n,j} \xrightarrow{(d)} X, \text{ with } \mathbb{P}\{X = d\} = 1$$

- **central region:** $j \sim \rho n$, with $0 < \rho < 1$:
  
  $R_{n,j}$ asymp. **binomial-distributed**
  
  $$R_{n,j} \xrightarrow{(d)} \text{Binomial}(d, 1 - \rho^{\frac{1}{d-1}})$$

- **j large:** $n - j = o(n)$:
  
  $$R_{n,j} \xrightarrow{(d)} X, \text{ with } \mathbb{P}\{X = 0\} = 1$$
**Results: Out-degree**

**Theorem (Kuba & Panholzer, 2007)**

Out-degree \( R_{n,j} \) of node \( j \) satisfies for \( d \)-ary trees:

- **j small:** \( j = o(n) \): asymp. \( j \) has **maximal out-degree** \( d \)

\[
R_{n,j} \xrightarrow{(d)} X, \text{ with } \mathbb{P}\{X = d\} = 1
\]

- **central region:** \( j \sim \rho n \), with \( 0 < \rho < 1 \):

\( R_{n,j} \) asymp. **binomial-distributed**

\[
R_{n,j} \xrightarrow{(d)} \text{Binomial}(d, 1 - \rho^{\frac{1}{d-1}})
\]

- **j large:** \( n - j = o(n) \):

\[
R_{n,j} \xrightarrow{(d)} X, \text{ with } \mathbb{P}\{X = 0\} = 1
\]
Results: Out-degree

Theorem (Kuba & Panholzer, 2007)

Out-degree $R_{n,j}$ of node $j$ satisfies for generalized plane-oriented recursive trees:

- **$j$ fixed**: $n^c R_{n,j}$ has limiting distribution
  
  - **Characterized via moments**: $n^c R_{n,j} \xrightarrow{(d)} X_j$, with
    
    $$\mathbb{E}(X_j^s) = \frac{\Gamma(s - 1 - c)\Gamma(j + c)}{\Gamma(-1 - c)\Gamma(j - (s - 1)c)}$$

  - **Characterized via density function**: $\frac{R_{n,j}}{\sqrt{n}} \xrightarrow{(d)} X_j$, with
    
    $$f_1(x) = \frac{-\Gamma(c)x^{-2-\frac{1}{c}}}{\Gamma(-1 - \frac{1}{c})\pi} \int_0^\infty e^{-(r^{\frac{1}{c}})r^{-\frac{1}{c}}-1}e^{-xr\cos(c\pi)}\sin(xr\sin(-c\pi))dr,$$
    
    $$f_j(x) = \frac{\Gamma(j + c)}{\Gamma(1 + c)\Gamma(j - 1)} \int_0^1 t^{2c}(1 - t)^{j-2}f_1(xt^c)dt, \quad \text{for } j \geq 2.$$
**Results: Out-degree**

**Theorem (Kuba & Panholzer, 2007)**

- **j small:** $j \to \infty$ such that $j = o(n)$:
  \[ n^c R_{n,j} \text{ asymp. Gamma-distributed} \]
  \[ n^c R_{n,j} \xrightarrow{(d)} \gamma\left(-1 - \frac{1}{c}\right) \]

- **Central region:** $j \sim \rho n$, with $0 < \rho < 1$:
  \[ R_{n,j} \text{ asymp. negativ binomial-distributed} \]
  \[ R_{n,j} \xrightarrow{(d)} \text{NegBin}\left(-1 - \frac{1}{c}, \rho^{-c}\right) \]

- **j large:** $n - j = o(n)$:
  \[ R_{n,j} \xrightarrow{(d)} X, \text{ with } \mathbb{P}\{X = 0\} = 1 \]
Results: Out-degree

Theorem (Kuba & Panholzer, 2007)

- **j small**: \( j \to \infty \) such that \( j = o(n) \):

  \[ n^c R_{n,j} \text{ asymp.} \gamma(-1 - \frac{1}{c}) \]

- **central region**: \( j \sim \rho n \), with \( 0 < \rho < 1 \):

  \[ R_{n,j} \text{ asymp. negativ binomial-distributed} \]

  \[ R_{n,j} \xrightarrow{(d)} \text{NegBin}(-1 - \frac{1}{c}, \rho^{-c}) \]

- **j large**: \( n - j = o(n) \):

  \[ R_{n,j} \xrightarrow{(d)} X \text{, with } \mathbb{P}\{X = 0\} = 1 \]
Results: Out-degree

Theorem (Kuba & Panholzer, 2007)

- **j small:** $j \to \infty$ such that $j = o(n)$:
  
  $n^c R_{n,j}$ asymp. **Gamma-distributed**

  \[ n^c R_{n,j} \xrightarrow{(d)} \gamma(-1 - \frac{1}{c}) \]

- **central region:** $j \sim \rho n$, with $0 < \rho < 1$:
  
  $R_{n,j}$ asymp. **negativ binomial-distributed**

  \[ R_{n,j} \xrightarrow{(d)} \text{NegBin}(-1 - \frac{1}{c}, \rho^{-c}) \]

- **j large:** $n - j = o(n)$:
  
  $R_{n,j} \xrightarrow{(d)} X$, with $\mathbb{P}\{X = 0\} = 1$
Results: Number of leaves

Theorem (Kuba & Panholzer, 2006)

Number of leaves $L_{n,j}$ in subtree rooted at $j$ satisfies:

- **$j$ fixed**: $L_{n,j}$ asympt.
  
  - **Recursive trees**: $\frac{2L_{n,j}}{n} \xrightarrow{(d)} \beta(1, j - 1)$,
  
  - **Binary incr. trees**: $\frac{3L_{n,j}}{n} \xrightarrow{(d)} \beta(2, j - 1)$,
  
  - **PORT**: $\frac{3L_{n,j}}{2n} \xrightarrow{(d)} \beta\left(\frac{1}{2}, j - 1\right)$

- **$j$ small**: $j \to \infty$ such that $j = o(n)$:
  
  $L_{n,j}$ asympt.

  - **Recursive trees**: $\frac{jL_{n,j}}{n} \xrightarrow{(d)} \gamma(1, 2)$,
  
  - **Binary incr. trees**: $\frac{jL_{n,j}}{n} \xrightarrow{(d)} \gamma(2, 3)$,
  
  - **PORT**: $\frac{jL_{n,j}}{n} \xrightarrow{(d)} \gamma\left(\frac{1}{2}, \frac{2}{3}\right)$
Results: Number of leaves

Theorem (Kuba & Panholzer, 2006)

Number of leaves $L_{n,j}$ in subtree rooted at $j$ satisfies:

- **$j$ fixed:** $L_{n,j}$ asympt. **Beta-distributed**
  - Recursive trees: $\frac{2L_{n,j}}{n} \xrightarrow{(d)} \beta(1, j - 1)$,
  - Binary incr. trees: $\frac{3L_{n,j}}{n} \xrightarrow{(d)} \beta(2, j - 1)$,
  - PORT: $\frac{3L_{n,j}}{2n} \xrightarrow{(d)} \beta(\frac{1}{2}, j - 1)$

- **$j$ small:** $j \to \infty$ such that $j = o(n)$:
  - $L_{n,j}$ asympt. **Gamma-distributed**
    - Recursive trees: $\frac{jL_{n,j}}{n} \xrightarrow{(d)} \gamma(1, 2)$,
    - Binary incr. trees: $\frac{jL_{n,j}}{n} \xrightarrow{(d)} \gamma(2, 3)$,
    - PORT: $\frac{jL_{n,j}}{n} \xrightarrow{(d)} \gamma(\frac{1}{2}, \frac{2}{3})$
Results: Number of leaves

**Theorem (Kuba & Panholzer, 2006)**

- **central region**: $j \sim \rho n$, with $0 < \rho < 1$:
  
  $L_{n,j}$ **has discrete limit law**

- **$j$ large**: $n - j = o(n)$:
  
  $$L_{n,j} \xrightarrow{(d)} X, \text{ with } \mathbb{P}\{X = 1\} = 1$$
Results: Number of leaves

Theorem (Kuba & Panholzer, 2006)

- **central region**: \( j \sim \rho n, \) with \( 0 < \rho < 1 \):
  
  \( L_{n,j} \) has **discrete limit law**

- **\( j \) large**: \( n - j = o(n) \):

  \[ L_{n,j} \xrightarrow{(d)} X, \text{ with } P\{X = 1\} = 1 \]
Proof
**Proof: Outline**

**Exact results:**

- **Decomposition of tree** into root node and subtrees

\[ T = T_1 T_2 T_r \]

\[ |T_i| = k_i \]

- **Recurrences for quantities considered**
  - **Depth:** reduce \( \mathbb{P}\{D_{n,j} = m\} \) to \( \mathbb{P}\{D_{k_1,i} = m - 1\} \)
  - **Subtree-size, out-degree, # leaves:** reduce \( \mathbb{P}\{X_{n,j} = m\} \) to \( \mathbb{P}\{X_{k_1,i} = m\} \)
  - **Distance:** reduce \( \mathbb{P}\{\Delta_{n,j} = m\} \) to \( \mathbb{P}\{\Delta_{k_1,i} = m\} \) and depth
Proof: Outline

Exact results:

- **Decomposition of tree** into root node and subtrees

\[ T = T_1 \cup T_2 \cup T_r \]
\[ |T_i| = k_i \]

- **Recurrences** for quantities considered
  - Depth: reduce \( P\{D_{n,j} = m\} \) to \( P\{D_{k_1,i} = m - 1\} \)
  - Subtree-size, out-degree, \# leaves: reduce \( P\{X_{n,j} = m\} \) to \( P\{X_{k_1,i} = m\} \)
  - Distance: reduce \( P\{\Delta_{n,j} = m\} \) to \( P\{\Delta_{k_1,i} = m\} \) and depth
Proof: Outline

- Introducing suitable trivariate generating functions
- Solving linear differential equations

Limiting distributions via different methods:
- Approximating closed form solution
- Studying characteristic function $\Rightarrow$ continuity theorem of Lévy
- Applying quasi-power theorem
- Using method of moments $\Rightarrow$ Theorem of Fréchet and Shohat
**Proof: Outline**

- Introducing suitable trivariate \textit{generating functions}
- Solving linear \textit{differential equations}

**Limiting distributions via different methods:**
- Approximating \textit{closed form solution}
- Studying \textit{characteristic function} $\Rightarrow$ continuity theorem of Lévy
- Applying \textit{quasi-power theorem}
- Using \textit{method of moments} $\Rightarrow$ Theorem of Fréchet and Shohat
Proof: Outline

- Introducing suitable trivariate generating functions
- Solving linear differential equations

Limiting distributions via different methods:

- Approximating closed form solution
- Studying characteristic function $\Rightarrow$ continuity theorem of Lévy
- Applying quasi-power theorem
- Using method of moments $\Rightarrow$ Theorem of Fréchet and Shohat
Extensions and generalizations
Extensions

Studies of further parameters

- Subtrees on the fringe [Kuba, 2006]
- Branching structure [Kuba, 2006]
- Weighted parameters [Kuba & Panholzer, 2007]
- Studies on most-recent common ancestor

Analyzing several nodes simultaneously

- Subtree-size [Kuba & Panholzer, 2009]

Analyzing several parameters simultaneously
Extensions

Studies of further parameters

- Subtrees on the fringe [Kuba, 2006]
- Branching structure [Kuba, 2006]
- Weighted parameters [Kuba & Panholzer, 2007]
- Studies on most-recent common ancestor

Analyzing several nodes simultaneously

- Subtree-size [Kuba & Panholzer, 2009]

Analyzing several parameters simultaneously
Extensions

Studies of further parameters
- Subtrees on the fringe [Kuba, 2006]
- Branching structure [Kuba, 2006]
- Weighted parameters [Kuba & Panholzer, 2007]
- Studies on most-recent common ancestor

Analyzing several nodes simultaneously
- Subtree-size [Kuba & Panholzer, 2009]

Analyzing several parameters simultaneously
Generalizations: Scale-free trees and $f$-trees

Analyzing trees obtained by slightly different growth rules:

- $f$-trees (Quintas)
- Scale-free trees (Mori, Katona)

Growth rule:

- Start with an edge labelled 1 – 2
- Attach new vertex to node with degree $d(v)$ proportional to:
  \[
  \begin{cases} 
  d(v) + \beta, & \text{scale-free trees} \\
  f - d(v), & \text{$f$-trees}
  \end{cases}
  \]
Generalizations: Scale-free trees and $f$-trees

Analyzing trees obtained by slightly different growth rules:

- $f$-trees (Quintas)
- Scale-free trees (Mori, Katona)

Growth rule:

- Start with an edge labelled $1 \rightarrow 2$
- Attach new vertex to node with degree $d(v)$ proportional to \[
\begin{cases} 
  d(v) + \beta, & \text{scale-free trees} \\
  f - d(v), & \text{$f$-trees}
\end{cases}
\]
Generalizations: Bucket recursive trees

Analyzing bucket recursive trees and generalizations:

**Bucket recursive trees** [Mahmoud and Smythe, 1995]:

- **Nodes are buckets**: can contain up to $b$ elements
- **Growth rule** for adding new vertex $n + 1$:
  - If node contains $c < b$ elements (unsaturated node): attracts new vertex with probability $\frac{c}{n}$; $n + 1$ added to this node
  - If node contains $b$ elements (saturated node): attracts new vertex with probability $\frac{c}{n}$; $n + 1$ attached to this node in a new bucket

Combinatorial model: “bucket increasing trees” [Kuba and Panholzer, 2008+]

- Analysis of parameters by extending recursive approach
Generalizations: Bucket recursive trees

Analyzing bucket recursive trees and generalizations:

Bucket recursive trees [Mahmoud and Smythe, 1995]:

- Nodes are buckets; can contain up to $b$ elements
- Growth rule for adding new vertex $n + 1$:
  - If node contains $c < b$ elements (unsaturated node): attracts new vertex with probability $\frac{c}{n}$; $n + 1$ added to this node
  - If node contains $b$ elements (saturated node): attracts new vertex with probability $\frac{c}{n}$; $n + 1$ attached to this node in a new bucket

Combinatorial model: “bucket increasing trees” [Kuba and Panholzer, 2008+]

- Analysis of parameters by extending recursive approach
**Generalizations:** Generalized Stirling permutations

**Analyzing generalized Stirling permutations:**

*\(k\)-Stirling permutations* (Gessel & Stanley, Park, Brenti):

- Permutations of the multiset \(\{1^k, 2^k, \ldots, n^k\}\)
- **Property:** elements occurring between two consecutive occurrences of \(i\) are larger than \(i\)

**Example:**

**Order \(n = 2\):** Three 2-Stirling permutations

\[1221, 1122, 2211.\]

**Note:** 1212, 2121, 2112 are NOT 2-Stirling permutations.

**Order \(n = 3\):** Fifteen 2-Stirling permutations

\[112233, 113322, 221133, 223311, 331122, 332211, 122133, 331221, 133122, 221331, 233211, 112332, 133221, 122331, 123321.\]
Generalizations: Generalized Stirling permutations

Analyzing generalized Stirling permutations:

\textit{k-Stirling permutations} (Gessel & Stanley, Park, Brenti):

- Permutations of the multiset \( \{1^k, 2^k, \ldots, n^k\} \)

- Property: elements occurring between two consecutive occurrences of \( i \) are larger than \( i \)

**Example:**

**Order \( n = 2 \):** Three 2-Stirling permutations

1221, 1122, 2211.

Note: 1212, 2121, 2112 are NOT 2-Stirling permutations.

**Order \( n = 3 \):** Fifteen 2-Stirling permutations

112233, 113322, 221133, 223311, 331122, 332211, 122133, 331221, 133122, 221331, 233211, 112332, 133221, 122331, 123321.
Generalizations: Generalized Stirling permutations

Analyzing generalized Stirling permutations:

$k$-Stirling permutations (Gessel & Stanley, Park, Brenti):

- Permutations of the multiset $\{1^k, 2^k, \ldots, n^k\}$
- Property: elements occurring between two consecutive occurrences of $i$ are larger than $i$

Example:

Order $n = 2$: Three $2$-Stirling permutations

1221, 1122, 2211.

Note: 1212, 2121, 2112 are NOT $2$-Stirling permutations.

Order $n = 3$: Fifteen $2$-Stirling permutations

112233, 113322, 221133, 223311, 331122, 332211, 122133, 331221, 133122, 221331, 233211, 112332, 133221, 122331, 123321.
Generalizations: Generalized Stirling permutations

Analyzing generalized Stirling permutations: $k$-Stirling permutations (Gessel & Stanley, Park, Brenti):

- Permutations of the multiset $\{1^k, 2^k, \ldots, n^k\}$
- Property: elements occurring between two consecutive occurrences of $i$ are larger than $i$

Example:

Order $n = 2$: Three 2-Stirling permutations

$1221, 1122, 2211.$

Note: $1212, 2121, 2112$ are NOT 2-Stirling permutations.

Order $n = 3$: Fifteen 2-Stirling permutations

$112233, 113322, 221133, 223311, 331122, 332211, 122133, 331221, 133122, 221331, 233211, 112332, 133221, 122331, 123321.$
Generalizations: Generalized Stirling permutations

Analyzing generalized Stirling permutations:

$k$-Stirling permutations (Gessel & Stanley, Park, Brenti):

- Permutations of the multiset $\{1^k, 2^k, \ldots, n^k\}$
- Property: elements occurring between two consecutive occurrences of $i$ are larger than $i$

Example:
Order $n = 2$: Three 2-Stirling permutations

1221, 1122, 2211.

Note: 1212, 2121, 2112 are NOT 2-Stirling permutations.

Order $n = 3$: Fifteen 2-Stirling permutations

112233, 113322, 221133, 223311, 331122, 332211, 122133, 331221, 133122, 221331, 233211, 112332, 133221, 122331, 123321.
Generalizations: Generalized Stirling permutations

Connection to increasing trees:

**Theorem (Gessel 78-94; Janson, Kuba & Panholzer 2008)**

*The class of $(k + 1)$-ary increasing trees of size $n$ is in bijection with the class of $k$-Stirling permutations of size $n.*

Proof uses depth first walk of trees:
**Connection to increasing trees:**

**Theorem (Gessel 78-94; Janson, Kuba & Panholzer 2008)**

*The class of $(k + 1)$-ary increasing trees of size $n$ is in bijection with the class of $k$-Stirling permutations of size $n.*

**Proof uses depth first walk of trees:**
Generalizations: Generalized Stirling permutations

Connection to increasing trees:

**Theorem (Gessel 78-94; Janson, Kuba & Panholzer 2008)**

*The class of \((k + 1)\)-ary increasing trees of size \(n\) is in bijection with the class of \(k\)-Stirling permutations of size \(n\).*

Proof uses depth first walk of trees:

```
  2211 1221 1122 233211
```