Embedding and Colouring
Odd Cycle Systems

Daniel Horsley and David Pike
Memorial University of Newfoundland
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Theorem (Alspach and Gavlas, 2001; Šajna, 2002):

An $m$-cycle system of order $v$ exists if and only if $v$ is odd, $v \geq m$, and $m$ divides $\binom{v}{2}$. 
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A weak $k$-colouring of a cycle system $S$ consists of a partition of the vertices of $S$ into $k$ colour classes such that no cycle of $S$ is monochromatic.

A cycle system $S$ is said to be $k$-chromatic if $k$ is the smallest integer for which $S$ admits a $k$-colouring.
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Example:

This system is 2-chromatic.
Some History – Triple Systems

- Every $\text{STS}(v)$ with $v \geq 7$ requires at least 3 colours. (Rosa and Pelikán, 1970)
- Every $\text{STS}(v)$ with $7 \leq v \leq 15$ is 3-chromatic. (Mathon, Phelps and Rosa, 1983)
- Every $\text{STS}(19)$ is 3-chromatic. (Forbes et al., 2009)
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• There is a 4-chromatic STS(21).  
  (Haddad, 1999)

• There is a 5-chromatic STS(63).  
  (Fugère, Haddad and Wehlau, 1994)

• There is a 6-chromatic STS(243).  
  (Bruen, Haddad and Wehlau, 1998)
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Theorem (Sotteau, 1981):

\( K_{a,b} \) can be decomposed into cycles of length \( 2t \) if and only if
\( a \equiv b \equiv 0 \pmod{2} \), \( a \geq t \), \( b \geq t \), and \( 2t \) divides \( ab \).
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Theorem (Horsley and Pike):
For each $k \geq 2$ and each odd $m \geq 5$, there exists a $k$-chromatic $m$-cycle system.
Definition:

A partial $m$-cycle system of order $u$ consists of a set of edge-disjoint $m$-cycles on $u$ vertices.

Example:

PTS(8)
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A (possibly partial) \( m \)-cycle system \( \mathcal{P} \) is said to be **embedded** in an \( m \)-cycle system \( \mathcal{S} \) if each cycle of \( \mathcal{P} \) is a cycle of \( \mathcal{S} \).
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Definition:

A (possibly partial) $m$-cycle system $P$ is said to be embedded in an $m$-cycle system $S$ if each cycle of $P$ is a cycle of $S$. 
Theorem (Erdős and Hajnal, 1966; Lovász, 1968):

Let $m \geq 2$, $k$ and $s$ be natural numbers. Then there exists a finite $m$-uniform set-system with chromatic number at least $k$ and with no circuit of length $s$ or less.
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For each $m \geq 3$, there exists a finite partial 2-$(v, m, 1)$ design with weak chromatic number at least $k$. 
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We now wish to show that weakly $k$-chromatic partial $m$-cycle systems can be embedded into weakly $k$-chromatic $m$-cycle systems.
To convey some of the flavour of such embeddings, we focus on $k = 3$ and $m = 5$.

Let $\mathcal{P}$ be a weakly 3-chromatic partial 5-cycle system of some order $u$, on vertex set $\mathbb{Z}_u$.

Let $\alpha : \mathbb{Z}_u \to \{c_1, c_2, c_3\}$ be a colouring of $\mathbb{Z}_u$ such that no cycle of $\mathcal{P}$ is monochromatic under $\alpha$.

Let $v \geq 10u + 7$ be 5-admissible (so $v \equiv 1$ or $5 \pmod{10}$). Let $t, w \in \mathbb{Z}$ such that $v = (2t + 1)(5) + w$ and $2 \leq w \leq 11$. Necessarily $w$ is even and $t \geq u$.

We will embed $\mathcal{P}$ in a 3-chromatic 5-cycle system $\mathcal{S}$ of order $v$, on vertex set $(\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W'$, where $|W| = |W'| = \frac{w}{2}$. 
\[ V = (\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W', \text{ where } |W| = |W'| = \frac{w}{2}. \]

Let \( \beta \) be a colouring of \( V \) such that:

- \( \beta(x) = c_1 \) for all \( x \in W \)
- \( \beta(x) = c_2 \) for all \( x \in W' \)
- \( \beta((x, i)) = \begin{cases} 
\alpha(x) & \text{if } x \in \mathbb{Z}_u \text{ and } i \in \{0, 2\} \\
\pi(\alpha(x)) & \text{if } x \in \mathbb{Z}_u \text{ and } i \in \{1, 3\} \\
\pi^2(\alpha(x)) & \text{if } x \in \mathbb{Z}_u \text{ and } i = 4 \\
c_1 & \text{if } x \in \mathbb{Z}_{2t+1} \setminus \mathbb{Z}_u \text{ and } i \in \{0, 2\} \\
c_2 & \text{if } x \in \mathbb{Z}_{2t+1} \setminus \mathbb{Z}_u \text{ and } i \in \{1, 3\} \\
c_3 & \text{if } x \in \mathbb{Z}_{2t+1} \setminus \mathbb{Z}_u \text{ and } i = 4 
\end{cases} \]

where \( \pi \) is the permutation \( (c_1, c_2, c_3) \).
\( V = (\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W', \text{ where } |W| = |W'| = \frac{w}{2}. \)
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Step 1 of 4

For each \( C = (x_0, x_1, x_2, x_3, x_4) \) in \( \mathcal{P} \)
there are vertices \( x_1 \) and \( x_3 \) with different colours.
\[ V = (\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W', \text{ where } |W| = |W'| = \frac{w}{2}. \]

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For each \( C = (x_0, x_1, x_2, x_3, x_4) \) in \( P \) there are vertices \( x_1 \) and \( x_3 \) with different colours.

Use a 5-cycle decomposition of \( C \cdot K^c_5 \) such that both:

1. each cycle contains both \((x_1, i)\) and \((x_3, i)\) for some \( i \)
2. the decomposition includes \((x_0, 0), (x_1, 0), (x_2, 0), (x_3, 0), (x_4, 0)\)

NB: \( P \) is embedded within \( \mathbb{Z}_u \times \{0\} \).
$V = (\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W'$, where $|W| = |W'| = \frac{w}{2}$.

Let $G$ be $K_{2t+1}$ with the edges of $\mathcal{P}$ removed.
\[ V = (\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W', \text{ where } |W| = |W'| = \frac{w}{2}. \]

Let \( G \) be \( K_{2t+1} \) with the edges of \( \mathcal{P} \) removed.

Decompose \( G \cdot K_5^c \) into 5-cycles such that, for each cycle \( C \), \((x, i)\) and \((x, i + 1)\) are in \( C \) for some \( x \in \mathbb{Z}_{2t+1} \) and some \( i \in \mathbb{Z}_5 \) (Lindner and Rodger, 1993).

All non-horizontal edges of \( \mathbb{Z}_{2t+1} \times \mathbb{Z}_5 \) are now used.
\[ V = (\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W', \text{ where } |W| = |W'| = \frac{w}{2}. \]

Let \( G \) be \( K_w^c \lor (K_5 \cup K_5) \), formed on \( W \cup W' \cup (\{2i, 2i+1\} \times \mathbb{Z}_5) \).
$V = (\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W'$, where $|W| = |W'| = \frac{w}{2}$.

Let $G$ be $K^c_w \lor (K_5 \cup K_5)$, formed on $W \cup W' \cup (\{2i, 2i + 1\} \times \mathbb{Z}_5)$.

Decompose $G$ into 5-cycles such that, for each cycle $C$, either

1. $V(C) \cap W \neq \emptyset$ and $V(C) \cap W' \neq \emptyset$

or

2. $V(C) \cap (W \cup W') = \emptyset$
\[ V = (\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W', \text{ where } |W| = |W'| = \frac{w}{2}. \]

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Do this for \( i = 0, \ldots, t - 1 \)
\[ V = (\mathbb{Z}_{2t+1} \times \mathbb{Z}_5) \cup W \cup W', \text{ where } |W| = |W'| = \frac{w}{2}. \]

Let \( G \) be \( K_{w+5} \) on \\
\( W \cup W' \cup (\{2t\} \times \mathbb{Z}_5) \).
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Step 4 of 4

Let \( G \) be \( K_{w+5} \) on 
\( W \cup W' \cup (\{2t\} \times \mathbb{Z}_5) \).

Decompose \( G \) into 5-cycles 
such that no cycle is monochromatic.

QED
Theorem (Horsley and Pike):

Let $k$ and $m$ be integers such that $k \geq 2$, $m \geq 3$ and $(k, m) \neq (2, 3)$. Then there is an integer $n'_{k,m}$ such that there exists a weakly $k$-chromatic $m$-cycle system of order $v$ for all $m$-admissible integers $v \geq n'_{k,m}$.

Theorem (Horsley and Pike):

Let $u_{k,m}$ be the minimum order of a weakly $k$-chromatic partial $m$-cycle system.

Let $n_{k,m}$ be the smallest $m$-admissible integer such that there exists a weakly $k$-chromatic $m$-cycle system of order $v$ for all $m$-admissible integers $v \geq n_{k,m}$.

Then $u_{k,m} \leq n_{k,m} \leq 2m(u_{k,m} + 1) + 1.$
Questions for Further Study:

- What about other types of designs?
- Perhaps a chromatic version of Wilson’s Theorem?

Acknowledgements: