Weak Near-Unanimity Functions and NP-Completeness Proofs

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The Homomorphism Problem.
- Complexity in the directed case: Unknown, only special cases e.g. semi-complete digraphs (Bang-Jensen, Hell and MacGillivray, 1988), smooth digraphs (Barto, Kozik and Niven 2008).

Weak Near-Unanimity Functions and NP-completeness.
- Conjecture of Bulatov, Jeavons and Krokhin.
- Gather evidence for this conjecture:
  - Some polynomial cases: $X$, $C_k$-extended $X$, graft extension.
  - NP-completeness: Indicator construction, vertex and arc sub-indicators.

WNUFs/no WNUFs for undirected graphs, semi-complete digraphs, and vertex transitive digraphs.
Let $G$ and $H$ be digraphs. A homomorphism from $G$ to $H$ is a mapping $f : V(G) \rightarrow V(H)$ such that $xy \in A(G)$ implies that $f(x)f(y) \in A(H)$. The existence of such a homomorphism is denoted by $G \rightarrow H$.

The homomorphism problem for the digraph $H$ is the problem of deciding for a given input $G$ whether $G \rightarrow H$. This is also known as the $H$-colouring problem, and is denoted by $\text{HOM}_H$. 
In the undirected case there is a dichotomy:

**Theorem (Hell and Nešetřil 1990)**

*Let $H$ be a graph with loops allowed.*

- *If $H$ is bipartite or contains a loop, then the $H$-colouring problem has a polynomial time algorithm.*
- *Otherwise the $H$-colouring problem is NP-complete.*
Complexity in the Directed Case

Only known in special cases:

**Theorem (Bang-Jensen, Hell and MacGillivray 1988)**

Let $H$ be a semi-complete digraph.

- If $H$ contains at most one directed cycle, then $H$-colouring is polynomial time solvable.
- Otherwise $H$-colouring is NP-complete.

**Theorem (Barto, Kozik and Niven 2008)**

Let $H$ be a digraph with no sources and no sinks.

- If the core of $H$ is a directed cycle, then $\text{HOM}_H$ is polynomial.
- If the core of $H$ is not a directed cycle, then $\text{HOM}_H$ is NP-complete.
The result by Barto, Kozik and Niven uses tools from universal algebra:

**Theorem (Bulatov, Jeavons and Krokhin; Larose and Zádori; Maróti and McKenzie)**

*If the digraph $H$ does not admit a weak near unanimity function, then $HOM_H$ is NP-complete.*

The existence of such a function is conjectured by Bulatov, Jeavons and Krokhin to determine the complexity of $HOM_H$ exactly:

**Conjecture**

*If the digraph $H$ admits a WNUF, then $HOM_H$ is polynomial, otherwise (if $H$ does not admit a WNUF) $HOM_H$ is NP-complete.*
A weak near-unanimity function \( f \) of arity \( k \) (\( \text{WNUF}_k \)) on the digraph \( H \) is

- **a polymorphism:** \( f : H^k \to H \),

\begin{align*}
V(H^k) &= \underbrace{V(H) \times V(H) \times \cdots \times V(H)}_{k} \\
(x_1, x_2, \ldots, x_k) \to (y_1, y_2, \ldots, y_k) \text{ iff } x_i \to y_i, \ 1 \leq i \leq k,
\end{align*}

- **idempotent:** \( f(x, x, \ldots, x) = x \),

- **weakly nearly-unanimous:**

\begin{align*}
f(y, x, x, \ldots, x, x) &= f(x, y, x, \ldots, x, x) = f(x, x, y, \ldots, x, x) = \\
&= \cdots = f(x, x, x, \ldots, x, y).
\end{align*}
Conjecture of Bulatov, Jeavons and Krokhin

Conjecture (Bulatov, Jeavons and Krokhin)

If $H$ admits a WNUF, then $HOM_H$ is polynomial time solvable. Otherwise $HOM_H$ is NP-complete (this part is known).

To gather evidence for this conjecture we could:

- Show that known algorithmic methods $\Rightarrow$ WNUFs.
- Try to show that digraphs $H$ for which $HOM_H$ is NP-complete, have no WNUF. Try to prove “no-WNUF” theorems.
- Prove it for various graph families.
  - Good candidates are ones where a dichotomy is known.
Theorem (Gutjahr, Woeginger and Welzl 1992)

Let $H$ be a digraph such that

- $H$ has an $X$-enumeration, or
- $H$ has the $C_k$-extended $X$ property, or
- $H$ is an instance of the graft extension.

Then $HOM_H$ is polynomial time solvable.

The goal here is to show that if a digraph $H$ has any of these properties, then it also has a WNUF.
The $X$-enumeration

Let $\{h_1, h_2, \ldots, h_n\}$ be an enumeration of the vertices of a digraph $H$. This enumeration is said to satisfy the $X$ property if $h_i h_j, h_k h_l \in A(H)$, implies that $\min(h_i, h_k) \min(h_j, h_l) \in A(H)$, where the minimum is taken with respect to the enumeration of $V(H)$. 

![Diagram](image)
The $X$-enumeration and WNUFs

**Theorem**

*If* $H$ has an $X$-enumeration, *then* $H$ has a WNUF$_2$.

**Proof:** Define $f : H^2 \to H$ by $f(x_1, x_2) = \min\{x_1, x_2\}$.

A converse is also true.

**Theorem**

*If there is an enumeration of $V(H)$ and a WNUF$_2$, $f : H^2 \to H$, such that for all $x_1, x_2 \in V(H)$, $f(x_1, x_2) = \min\{x_1, x_2\}$, then this enumeration is an $X$-enumeration.*

**Proof:** Suppose $u_1u_2$ and $v_1v_2 \in E(H)$. Then $(u_1, v_1)(u_2, v_2) \in E(H \times H)$, so $f(u_1, v_1) = \min\{u_1, v_1\}$ is adjacent to $f(u_2, v_2) = \min\{u_2, v_2\}$ in $H$.

Arity 2 can be replaced by arity $k$. 

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WNUFs and NP-c
The $C_k$-extended $X$ Property

\[ V_0 \rightarrow V_k - 1 \rightarrow \ldots \rightarrow V_1 \rightarrow V_3 \rightarrow V_2 \]

\[ \ldots \]

\[ V_0 \rightarrow X \rightarrow V_1 \]
The $X$-graft Extension

- $h_1, h_2, \ldots, h_n$ is an $X$-enumeration $H_1$.
- $H_2$ is another digraph.

Form $H$ by replacing $h_n$ by $H_2$ as in the wreath product. The digraph $H$ is called $\text{graft}(H_1, H_2)$. 
**Theorem**

Let $H$ be a digraph such that

- $H$ has an $X$-enumeration, or
- $H$ has the $C_k$-extended $X$ property, or
- $H = \text{graft}(H_1, H_2)$, where $H_2$ has a WNUF$_k$.

Then $H$ has a WNUF$_k$. 

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WNUFs and NP-c
Four main tools in proving NP-completeness of $\text{HOM}_H$:

- Direct reduction from SAT or similar.
- The “indicator” construction.
- The “(vertex) sub-indicator” construction.
- The “arc sub-indicator” construction.

For digraphs $H$ in some family, the proofs normally go by induction on $|V(H)|$ (non-existence of minimum counterexample). The direct reductions establish the base cases and the three constructions make the induction work.
Identify $u$ and $v$, retract $J$ back to $T_5$, and keep track of the images of $j$. 
In general, some designated vertices of the sub-indicator $J$ are identified with some designated vertices of $H$. Let $H^+$ be the subgraph of $H$ induced by the images of $j$ under retraction back to $H$.

**Lemma (Hell and Nešetřil 1990)**

*Let $H$ be a digraph that is a core. If the $H^+$-colouring problem is NP-complete, then the $H$-colouring problem is also NP-complete.*

**Lemma**

*Let $H$ be a digraph. If $H^+$ does not have a WNUF$_k$ for $k > 1$, then $H$ does not have a WNUF of arity greater than one.*

The proof shows the contrapositive.

There are similar results corresponding to the indicator construction and arc sub-indicator construction.
These results give a method for translating certain NP-completeness theorems into no WNUF theorems.

- “NP-complete” $\iff$ “no WNUF.”
- If there is a proof of NP-completeness that depends on base cases $B_1, B_2, \ldots, B_t$ and the application of vertex (arc) sub-indicators and indicators, then there is a proof of “no WNUF” provided one can show that $B_1, B_2, \ldots, B_t$ have no WNUF.
Theorem (Bang-Jensen, Hell and MacGillivray 1988)

Let $H$ be a semi-complete digraph.

- If $H$ contains at most one directed cycle, then $H$-colouring is polynomial time solvable.
- Otherwise $H$-colouring is NP-complete.

- Acyclic tournaments have an $X$-enumeration $\Rightarrow$ have a WNUF.
- Unicyclic tournaments are instances of the graft extension $\Rightarrow$ have a WNUF.
Semi-complete Digraphs With At Least Two Cycles

Base Cases:

\[ T_4 \]

\[ T'_4 \]

\[ T_5 \]

\[ T_6 \]
Theorem

Let $H$ be a semi-complete digraph.

- If $H$ contains at most one directed cycle, then $H$ has a WNUF.
- Otherwise $H$ does not admit a WNUF.
Undirected Graphs and WNUFs

Theorem (Hell and Nešetřil 1990)

Let $H$ be a core.

- If $H$ is bipartite, then the $H$-colouring problem has a polynomial time algorithm.
- Otherwise the $H$-colouring problem is NP-complete.

Theorem

Let $H$ be a core.

- If $H = K_2$, then $H$ admits a NUF$_3$.
- If $H$ is non-bipartite, then $H$ does not admit a WNUF.

Base case is $K_3$. 
Theorem (MacGillivray 1991)

Let $H$ be a vertex transitive digraph that is also a core.
- If $H = C_k$, $\text{HOM}_H$ is polynomial.
- Otherwise, $\text{HOM}_H$ is NP-complete.

Theorem

Let $H$ be a vertex transitive digraph that is also a core.
- If $H = C_k$, then $H$ admits a NUF\textsubscript{3}.
- Otherwise, $H$ does not admit a WNUF.

Base Cases: Undirected non-bipartite graphs.
Thank you.
Questions?