



Soliton solutions to quadrilateral lattice equations

Jarmo Hietarinta

Department of Physics, University of Turku, FIN-20014 Turku, Finland

in collaboration with
James Atkinson, Frank Nijhoff and Da-jun Zhang

Symmetries and Integrability of Difference Equations
Sainte-Adele, Quebec, 22-26.6.2008



The setting

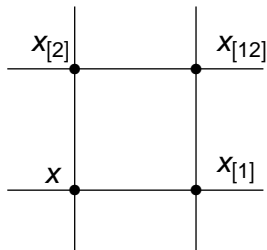
We consider lattice maps defined on an elementary square:

$$x_{n,m} = x_{00} = x$$

$$x_{n+1,m} = x_{10} = x_{[1]} = \tilde{x}$$

$$x_{n,m+1} = x_{01} = x_{[2]} = \hat{x}$$

$$x_{n+1,m+1} = x_{11} = x_{[12]} = \hat{\tilde{x}}$$



The setting

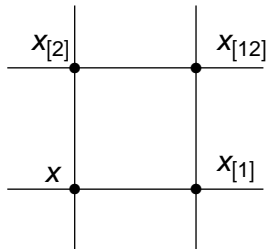
We consider lattice maps defined on an elementary square:

$$x_{n,m} = x_{00} = x$$

$$x_{n+1,m} = x_{10} = x_{[1]} = \tilde{x}$$

$$x_{n,m+1} = x_{01} = x_{[2]} = \hat{x}$$

$$x_{n+1,m+1} = x_{11} = x_{[12]} = \widehat{\tilde{x}}$$



The four corner values are related by a multi-linear equation:

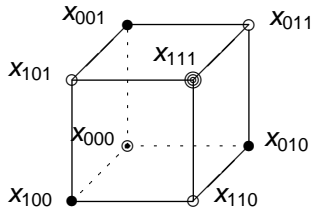
$$Q(x, x_{[1]}, x_{[2]}, x_{[12]}; p, q) = 0$$

This allows propagation from initial data given on a staircase or on a corner.

CAC - Consistency Around a Cube

One definition of integrability for such lattices is that they should allow consistent extensions from 2D to 3D (and higher)

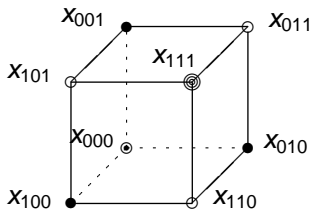
Adjoin a third direction $x_{n,m} \rightarrow x_{n,m,k}$ and construct a cube.



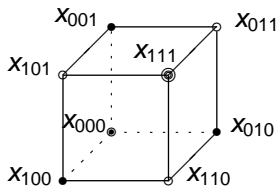
CAC - Consistency Around a Cube

One definition of integrability for such lattices is that they should allow consistent extensions from 2D to 3D (and higher)

Adjoin a third direction $x_{n,m} \rightarrow x_{n,m,k}$ and construct a cube.



Map at the bottom $Q_{12}(x, \tilde{x}, \hat{x}, \tilde{\hat{x}}; p, q) = 0$,
on the sides $Q_{23}(x, \hat{x}, \bar{x}, \tilde{\bar{x}}; q, r) = 0$, $Q_{31}(x, \bar{x}, \tilde{x}, \tilde{\tilde{x}}; r, p) = 0$,
shifted maps on parallel shifted planes.

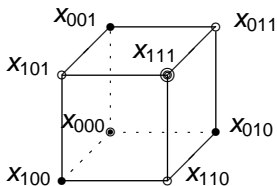


Consistency problem:

Given values at black disks, we can
compute values at open disks uniquely.

But x_{111} can be computed in 3 different ways!

They must agree!



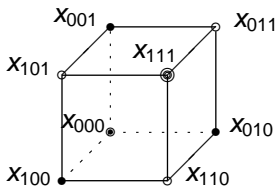
Consistency problem:

Given values at black disks, we can
compute values at open disks uniquely.

But x_{111} can be computed in 3 different ways!

They must agree!

- CAC represents a rather high level of integrability.



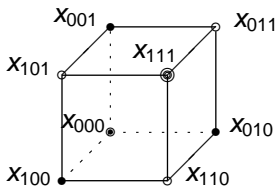
Consistency problem:

Given values at black disks, we can compute values at open disks uniquely.

But x_{111} can be computed in 3 different ways!

They must agree!

- CAC represents a rather high level of integrability.
- It is a kind of Bianchi identity [Nimmo and Schief, Proc. R. Soc. Lond. A **453** (1997) 255].



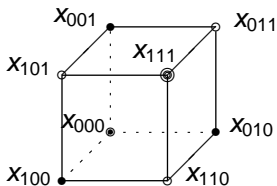
Consistency problem:

Given values at black disks, we can compute values at open disks uniquely.

But x_{111} can be computed in 3 different ways!

They must agree!

- CAC represents a rather high level of integrability.
- It is a kind of Bianchi identity [Nimmo and Schief, Proc. R. Soc. Lond. A **453** (1997) 255].
- It allows construction of Lax presentation [Nijhoff and Walker, Glasgow Math. J. **43A** (2001) 109].



Consistency problem:

Given values at black disks, we can compute values at open disks uniquely.

But x_{111} can be computed in 3 different ways!

They must agree!

- CAC represents a rather high level of integrability.
- It is a kind of Bianchi identity [Nimmo and Schief, Proc. R. Soc. Lond. A **453** (1997) 255].
- It allows construction of Lax presentation [Nijhoff and Walker, Glasgow Math. J. **43A** (2001) 109].
- It can be used as a method to classify integrable equation. [Adler, Bobenko and Suris, Commun. Math. Phys. **233** 513 (2003)].

ABS classification

Assume CAC with 2 additional assumptions:

- symmetry ($\varepsilon, \sigma = \pm 1$):

$$\begin{aligned} Q(x_{000}, x_{100}, x_{010}, x_{110}; p, q) &= \varepsilon Q(x_{000}, x_{010}, x_{100}, x_{110}; q, p) \\ &= \sigma Q(x_{100}, x_{000}, x_{110}, x_{010}; p, q) \end{aligned}$$

ABS classification

Assume CAC with 2 additional assumptions:

- symmetry ($\varepsilon, \sigma = \pm 1$):

$$\begin{aligned} Q(x_{000}, x_{100}, x_{010}, x_{110}; p, q) &= \varepsilon Q(x_{000}, x_{010}, x_{100}, x_{110}; q, p) \\ &= \sigma Q(x_{100}, x_{000}, x_{110}, x_{010}; p, q) \end{aligned}$$

- “tetrahedron property”: x_{111} does not depend on x_{000} .

ABS classification

Assume CAC with 2 additional assumptions:

- symmetry ($\varepsilon, \sigma = \pm 1$):

$$\begin{aligned} Q(x_{000}, x_{100}, x_{010}, x_{110}; p, q) &= \varepsilon Q(x_{000}, x_{010}, x_{100}, x_{110}; q, p) \\ &= \sigma Q(x_{100}, x_{000}, x_{110}, x_{010}; p, q) \end{aligned}$$

- “tetrahedron property”: x_{111} does not depend on x_{000} .

Result: complete classification under these assumptions,
9 models.

ABS results:

List H :

$$(H1) \quad (x - \hat{x})(\tilde{x} - \hat{x}) + q - p = 0,$$

$$(H2) \quad (x - \hat{x})(\tilde{x} - \hat{x}) + (q - p)(x + \tilde{x} + \hat{x} + \hat{\hat{x}}) + q^2 - p^2 = 0,$$

$$(H3) \quad p(x\tilde{x} + \hat{x}\hat{\hat{x}}) - q(x\hat{x} + \tilde{x}\hat{\hat{x}}) + \delta(p^2 - q^2) = 0.$$

List A :

$$(A1) \quad p(x + \hat{x})(\tilde{x} + \hat{\hat{x}}) - q(x + \tilde{x})(\hat{x} + \hat{\hat{x}}) - \delta^2 pq(p - q) = 0,$$

(A2)

$$(q^2 - p^2)(x\tilde{x}\hat{x}\hat{\hat{x}} + 1) + q(p^2 - 1)(x\hat{x} + \tilde{x}\hat{\hat{x}}) - p(q^2 - 1)(x\tilde{x} + \hat{x}\hat{\hat{x}}) = 0.$$

Main list:

$$(Q1) \quad p(x - \hat{x})(\tilde{x} - \hat{\tilde{x}}) - q(x - \tilde{x})(\hat{x} - \hat{\tilde{x}}) + \delta^2 pq(p - q) = 0,$$

(Q2)

$$p(x - \hat{x})(\tilde{x} - \hat{\tilde{x}}) - q(x - \tilde{x})(\hat{x} - \hat{\tilde{x}}) + pq(p - q)(x + \tilde{x} + \hat{x} + \hat{\tilde{x}}) - pq(p - q)(p^2 - pq + q^2) = 0,$$

(Q3)

$$(q^2 - p^2)(x\hat{\tilde{x}} + \tilde{x}\hat{x}) + q(p^2 - 1)(x\tilde{x} + \hat{x}\hat{\tilde{x}}) - p(q^2 - 1)(x\hat{x} + \tilde{x}\hat{\tilde{x}}) - \delta^2(p^2 - q^2)(p^2 - 1)(q^2 - 1)/(4pq) = 0,$$

(Q4) (the root model from which others follow)

$$a_0 x\tilde{x}\hat{x}\hat{\tilde{x}} + a_1(x\tilde{x}\hat{x} + \tilde{x}\hat{x}\hat{\tilde{x}} + \hat{x}\hat{\tilde{x}}x + \hat{\tilde{x}}x\tilde{x}) + a_2(x\hat{\tilde{x}} + \tilde{x}\hat{x}) + \bar{a}_2(x\tilde{x} + \hat{x}\hat{\tilde{x}}) + \tilde{a}_2(x\hat{x} + \tilde{x}\hat{\tilde{x}}) + a_3(x + \tilde{x} + \hat{x} + \hat{\tilde{x}}) + a_4 = 0,$$

where the a_i depend on the lattice directions and are given in terms of Weierstrass elliptic functions. This was first derived by Adler as a superposition rule of BT's for the Krichever-Novikov equation. [Adler, Intl. Math. Res. Notices, # 1 (1998) 1-4]

J.H., JNMP **12** Suppl 2, 223 (2005). a simpler Jacobi form for (Q4) of ABS:

$$(h_1 f_2 - h_2 f_1)[(xx_{[1]}x_{[12]}x_{[2]} + 1)f_1 f_2 - (xx_{[12]} + x_{[1]}x_{[2]})] \\ + (f_1^2 f_2^2 - 1)[(xx_{[1]} + x_{[12]}x_{[2]})f_1 - (xx_{[2]} + x_{[1]}x_{[12]})f_2] = 0,$$

$h_i^2 = f_i^4 + \delta f_i^2 + 1$, parametrizable by Jacobi elliptic functions.

Hirota's bilinear method

Recall Hirota's direct method for constructing multisoliton solutions in the continuous case:

Hirota's bilinear method

Recall Hirota's direct method for constructing multisoliton solutions in the continuous case:

- 1 find a **dependent variable transformation** into Hirota bilinear form

Hirota's bilinear method

Recall Hirota's direct method for constructing multisoliton solutions in the continuous case:

- 1 find a **dependent variable transformation** into Hirota bilinear form
 - 1 find background or vacuum solutions

Hirota's bilinear method

Recall Hirota's direct method for constructing multisoliton solutions in the continuous case:

- 1 find a **dependent variable transformation** into Hirota bilinear form
 - 1 find background or vacuum solutions
 - 2 find a 1-soliton-solutions (1SS)

Hirota's bilinear method

Recall Hirota's direct method for constructing multisoliton solutions in the continuous case:

- 1 find a **dependent variable transformation** into Hirota bilinear form
 - 1 find background or vacuum solutions
 - 2 find a 1-soliton-solutions (1SS)
 - 3 use this info to guess the transformation

Hirota's bilinear method

Recall Hirota's direct method for constructing multisoliton solutions in the continuous case:

- 1 find a **dependent variable transformation** into Hirota bilinear form
 - 1 find background or vacuum solutions
 - 2 find a 1-soliton-solutions (1SS)
 - 3 use this info to guess the transformation
- 2 construct the first few soliton solutions perturbatively

Hirota's bilinear method

Recall Hirota's direct method for constructing multisoliton solutions in the continuous case:

- 1 find a **dependent variable transformation** into Hirota bilinear form
 - 1 find background or vacuum solutions
 - 2 find a 1-soliton-solutions (1SS)
 - 3 use this info to guess the transformation
- 2 construct the first few soliton solutions perturbatively
- 3 guess the general form (usually a determinant: Wronskian, Pfaffian etc) and prove it

Hirota's bilinear method

Recall Hirota's direct method for constructing multisoliton solutions in the continuous case:

- 1 find a **dependent variable transformation** into Hirota bilinear form
 - 1 find background or vacuum solutions
 - 2 find a 1-soliton-solutions (1SS)
 - 3 use this info to guess the transformation
- 2 construct the first few soliton solutions perturbatively
- 3 guess the general form (usually a determinant: Wronskian, Pfaffian etc) and prove it

Hirota's bilinear form is well suited for constructing soliton solutions, because the new dependent variable is a **polynomial of exponentials with linear exponents**.

What is a discrete Hirota bilinear form?

The key property of an equation in Hirota's bilinear form is invariance under a gauge transformation

$$f_i \rightarrow f'_i := e^{ax+bt} f_i.$$

or in the discrete case

$$f_j(n, m) \rightarrow f'_j(n, m) = A^n B^m f_j(n, m).$$

What is a discrete Hirota bilinear form?

The key property of an equation in Hirota's bilinear form is invariance under a gauge transformation

$$f_i \rightarrow f'_i := e^{ax+bt} f_i.$$

or in the discrete case

$$f_j(n, m) \rightarrow f'_j(n, m) = A^n B^m f_j(n, m).$$

We say an equation is in **Hirota bilinear (HB) form** if it can be written as

$$f_j(n, m) \rightarrow \sum_j c_j f_j(n + \nu_j^+, m + \mu_j^+) g_j(n + \nu_j^-, m + \mu_j^-) = 0$$

where $\nu_i^+ + \nu_i^- = \nu_k^+ + \nu_k^-$, $\mu_i^+ + \mu_i^- = \mu_k^+ + \mu_k^-$, $\forall i, k$,
that is, the equation is a sum of quadratic terms, and the index sums are the same in each component of the sum.

The background solution

First problem in the perturbative approach:

What is the background solution?

The background solution

First problem in the perturbative approach:

What is the background solution?

Atkinson: Take the CAC cube and insist that the solution is a **fixed point** of the bar shift. The “side”-equations are then

$$Q(u, \tilde{u}, u, \tilde{u}; p, r) = 0, \quad Q(u, \hat{u}, u, \hat{u}; q, r) = 0.$$

The background solution

First problem in the perturbative approach:

What is the background solution?

Atkinson: Take the CAC cube and insist that the solution is a **fixed point** of the bar shift. The “side”-equations are then

$$Q(u, \tilde{u}, u, \tilde{u}; p, r) = 0, \quad Q(u, \hat{u}, u, \hat{u}; q, r) = 0.$$

The H1 equation is given by

$$\text{H1} \equiv (u - \hat{\tilde{u}})(\tilde{u} - \hat{u}) - (p - q) = 0,$$

then the side-equations are

$$(\tilde{u} - u)^2 = r - p, \quad (\hat{u} - u)^2 = r - q.$$

The background solution

First problem in the perturbative approach:

What is the background solution?

Atkinson: Take the CAC cube and insist that the solution is a **fixed point** of the bar shift. The “side”-equations are then

$$Q(u, \tilde{u}, u, \tilde{u}; p, r) = 0, \quad Q(u, \hat{u}, u, \hat{u}; q, r) = 0.$$

The H1 equation is given by

$$\text{H1} \equiv (u - \hat{u})(\tilde{u} - \hat{u}) - (p - q) = 0,$$

then the side-equations are

$$(\tilde{u} - u)^2 = r - p, \quad (\hat{u} - u)^2 = r - q.$$

For convenience we reparametrize $(p, q) \rightarrow (a, b)$ by

$$p = r - a^2, \quad q = r - b^2.$$

The side-equations then factorize as

$$(\tilde{u} - u - a)(\tilde{u} - u + a) = 0, \quad (\hat{u} - u - b)(\hat{u} - u + b) = 0,$$

The side-equations then factorize as

$$(\tilde{u} - u - a)(\tilde{u} - u + a) = 0, \quad (\hat{u} - u - b)(\hat{u} - u + b) = 0,$$

Since the factor that vanishes may depend on n, m we actually have to solve

$$\tilde{u} - u = (-1)^\theta a, \quad \hat{u} - u = (-1)^\chi b,$$

where $\theta, \chi \in \mathbb{Z}$ may depend on n, m .

The side-equations then factorize as

$$(\tilde{u} - u - a)(\tilde{u} - u + a) = 0, \quad (\hat{u} - u - b)(\hat{u} - u + b) = 0,$$

Since the factor that vanishes may depend on n, m we actually have to solve

$$\tilde{u} - u = (-1)^\theta a, \quad \hat{u} - u = (-1)^\chi b,$$

where $\theta, \chi \in \mathbb{Z}$ may depend on n, m .

From consistency $\theta, \in \{n, 0\}, \chi, \in \{m, 0\}$.

The side-equations then factorize as

$$(\tilde{u} - u - a)(\tilde{u} - u + a) = 0, \quad (\hat{u} - u - b)(\hat{u} - u + b) = 0,$$

Since the factor that vanishes may depend on n, m we actually have to solve

$$\tilde{u} - u = (-1)^\theta a, \quad \hat{u} - u = (-1)^\chi b,$$

where $\theta, \chi \in \mathbb{Z}$ may depend on n, m .

From consistency $\theta, \in \{n, 0\}, \chi, \in \{m, 0\}$.

The set of possible background solution turns out to be

$$\begin{aligned} & an + bm + \gamma, \\ & \frac{1}{2}(-1)^n a + bm + \gamma, \\ & an + \frac{1}{2}(-1)^m b + \gamma, \\ & \frac{1}{2}(-1)^n a + \frac{1}{2}(-1)^m b + \gamma. \end{aligned}$$

1SS

[Atkinson, JH and Nijhoff, JPhysA **41** (2008) 142001.]

The BT generating 1SS for H1 is

$$(u - \tilde{u})(\tilde{u} - \bar{u}) = p - \kappa,$$

$$(u - \hat{u})(\bar{u} - \hat{u}) = \kappa - q.$$

- u is the background solution $an + bm + \gamma$,
- \bar{u} is the new 1SS,
- κ is the soliton parameter (the parameter in the bar-direction).

1SS

[Atkinson, JH and Nijhoff, JPhysA **41** (2008) 142001.]

The BT generating 1SS for H1 is

$$(u - \tilde{u})(\tilde{u} - \bar{u}) = p - \kappa,$$

$$(u - \hat{u})(\bar{u} - \hat{u}) = \kappa - q.$$

- u is the background solution $an + bm + \gamma$,
- \bar{u} is the new 1SS,
- κ is the soliton parameter (the parameter in the bar-direction).

We search for a new solution \bar{u} of the form

$$\bar{u} = \bar{u}_0 + v,$$

where \bar{u}_0 is the bar-shifted background solution

$$\bar{u}_0 = an + bm + k + \lambda.$$

For v we then find (in the case of H1)

$$\tilde{v} = \frac{Ev}{v + F}, \quad \hat{v} = \frac{Gv}{v + H},$$

where

$$E = -(a+k), \quad F = -(a-k), \quad G = -(b+k), \quad H = -(b-k),$$

and k is related to \varkappa by $\varkappa = r - k^2$.

For v we then find (in the case of H1)

$$\tilde{v} = \frac{Ev}{v+F}, \quad \hat{v} = \frac{Gv}{v+H},$$

where

$$E = -(a+k), \quad F = -(a-k), \quad G = -(b+k), \quad H = -(b-k),$$

and k is related to \varkappa by $\varkappa = r - k^2$.

Introducing $v = f/g$ and $\Phi = (g, f)^T$ we can write the v -equations as matrix equations

$$\Phi(n+1, m) = \mathcal{N}(n, m)\Phi(n, m), \quad \Phi(n, m+1) = \mathcal{M}(n, m)\Phi(n, m),$$

where

$$\mathcal{N}(n, m) = \Lambda \begin{pmatrix} E & 0 \\ 1 & F \end{pmatrix}, \quad \mathcal{M}(n, m) = \Lambda' \begin{pmatrix} G & 0 \\ 1 & H \end{pmatrix},$$

In this case E, F, G, H are constants and we can choose $\Lambda = \Lambda' = 1$.

Since the matrices \mathcal{N} , \mathcal{M} commute it is easy to find

$$\Phi(n, m) = \begin{pmatrix} E^n G^m & 0 \\ \frac{E^n G^m - F^n H^m}{-2k} & F^n H^m \end{pmatrix} \Phi(0, 0).$$

Since the matrices \mathcal{N}, \mathcal{M} commute it is easy to find

$$\Phi(n, m) = \begin{pmatrix} E^n G^m & 0 \\ \frac{E^n G^m - F^n H^m}{-2k} & F^n H^m \end{pmatrix} \Phi(0, 0).$$

If we let

$$\rho_{n,m} = \left(\frac{E}{F} \right)^n \left(\frac{G}{H} \right)^m \rho_{0,0} = \left(\frac{a+k}{a-k} \right)^n \left(\frac{b+k}{b-k} \right)^m \rho_{0,0},$$

(ρ takes the role of e^η)

Since the matrices \mathcal{N} , \mathcal{M} commute it is easy to find

$$\Phi(n, m) = \begin{pmatrix} E^n G^m & 0 \\ \frac{E^n G^m - F^n H^m}{-2k} & F^n H^m \end{pmatrix} \Phi(0, 0).$$

If we let

$$\rho_{n,m} = \left(\frac{E}{F}\right)^n \left(\frac{G}{H}\right)^m \rho_{0,0} = \left(\frac{a+k}{a-k}\right)^n \left(\frac{b+k}{b-k}\right)^m \rho_{0,0},$$

(ρ takes the role of e^η) then we obtain

$$v_{n,m} = \frac{-2k\rho_{n,m}}{1 + \rho_{n,m}}.$$

Finally we obtain the 1SS for H1:

$$u_{n,m}^{(1SS)} = (an + bm + \lambda) + k + \frac{-2k\rho_{n,m}}{1 + \rho_{n,m}}.$$

Bilinearizing transformation

In an explicit form the 1SS is

$$u_{n,m}^{1SS} = an + bm + \lambda + \frac{k(1-\rho_{n,m})}{1+\rho_{n,m}}$$

Bilinearizing transformation

In an explicit form the 1SS is

$$u_{n,m}^{1SS} = an + bm + \lambda + \frac{k(1-\rho_{n,m})}{1+\rho_{n,m}}$$

This suggests the dependent variable transformation

$$u_{n,m} = an + bm + \lambda - \frac{g_{n,m}}{f_{n,m}}.$$

Bilinearizing transformation

In an explicit form the 1SS is

$$u_{n,m}^{1SS} = an + bm + \lambda + \frac{k(1-\rho_{n,m})}{1+\rho_{n,m}}$$

This suggests the dependent variable transformation

$$u_{n,m} = an + bm + \lambda - \frac{g_{n,m}}{f_{n,m}}.$$

Then it is easy to show that

$$\begin{aligned} H1 &\equiv (u - \widehat{u})(\widetilde{u} - \widehat{u}) - p + q \\ &= -[\mathcal{H}_1 + (a-b)\widehat{f\widetilde{f}}][\mathcal{H}_2 + (a+b)\widetilde{f\widehat{f}}]/(\widetilde{f\widehat{f\widehat{f}}}) + (a^2 - b^2), \end{aligned}$$

where

$$\begin{aligned} \mathcal{H}_1 &\equiv \widehat{g\widetilde{f}} - \widetilde{g\widehat{f}} + (a-b)(\widetilde{f\widehat{f}} - \widehat{f\widetilde{f}}) = 0, \\ \mathcal{H}_2 &\equiv \widetilde{g\widehat{f}} - \widehat{g\widetilde{f}} + (a+b)(\widehat{f\widetilde{f}} - \widetilde{f\widehat{f}}) = 0. \end{aligned}$$

Casoratians

For given functions $\varphi_i(n, m, h)$ we define the column vectors

$$\varphi(n, m, h) = (\varphi_1(n, m, h), \varphi_2(n, m, h), \dots, \varphi_N(n, m, h))^T,$$

Casoratians

For given functions $\varphi_i(n, m, h)$ we define the column vectors

$$\varphi(n, m, h) = (\varphi_1(n, m, h), \varphi_2(n, m, h), \dots, \varphi_N(n, m, h))^T,$$

and then compose the $N \times N$ Casorati matrix from columns with different shifts h_i , and then the determinant

$$\mathbf{C}_{n,m}(\varphi; \{h_i\}) = |\varphi(n, m, h_1), \varphi(n, m, h_2), \dots, \varphi(n, m, h_N)|.$$

Casoratians

For given functions $\varphi_i(n, m, h)$ we define the column vectors

$$\varphi(n, m, h) = (\varphi_1(n, m, h), \varphi_2(n, m, h), \dots, \varphi_N(n, m, h))^T,$$

and then compose the $N \times N$ Casorati matrix from columns with different shifts h_i , and then the determinant

$$C_{n,m}(\varphi; \{h_i\}) = |\varphi(n, m, h_1), \varphi(n, m, h_2), \dots, \varphi(n, m, h_N)|.$$

For example

$$\begin{aligned} C_{n,m}^1(\varphi) &:= |\varphi(n, m, 0), \varphi(n, m, 1), \dots, \varphi(n, m, N-1)| \\ &\equiv |0, 1, \dots, N-1| \equiv |\widehat{N-1}|, \\ C_{n,m}^2(\varphi) &:= |\varphi(n, m, 0), \dots, \varphi(n, m, N-2), \varphi(n, m, N)| \\ &\equiv |0, 1, \dots, N-2, N| \equiv |\widehat{N-2}, N|, \end{aligned}$$

Main result

The bilinear equations \mathcal{H}_i are solved by Casoratians

$f = |\widehat{N-1}|$, $g = |\widehat{N-2}, N|$, with ψ given by

$$\psi_i(n, m, h; k_i) = \varrho_i^+ k_i^h (a+k_i)^n (b+k_i)^m + \varrho_i^- (-k_i)^h (a-k_i)^n (b-k_i)^m.$$

Main result

The bilinear equations \mathcal{H}_i are solved by Casoratians

$f = |\widehat{N-1}|$, $g = |\widehat{N-2, N}|$, with ψ given by

$$\psi_i(n, m, h; k_i) = \varrho_i^+ k_i^h (a+k_i)^n (b+k_i)^m + \varrho_i^- (-k_i)^h (a-k_i)^n (b-k_i)^m.$$

Similar results exist for H2, H3, A1, Q1

[JH and Da-jun Zhang, to appear],

and for Q3 [Atkinson, JH and Nijhoff]

The structure of the soliton solution is similar to those of the Hirota-Miwa equation

Conclusions

We have considered the construction of multisoliton solution for the quadrilateral lattice equations in the ABS list.

A detailed straightforward construction exists and has been done for the low members in the ABS list (H1,H2,H3,A1,Q1).
(JH and Da-jun Zhang, work in progress)

Conclusions

We have considered the construction of multisoliton solution for the quadrilateral lattice equations in the ABS list.

A detailed straightforward construction exists has been done for the low members in the ABS list (H1,H2,H3,A1,Q1).
(JH and Da-jun Zhang, work in progress)

Explicit NSS has also been derived for Q3, using a connection to the NQC equation.
(Atkinson, JH and Nijhoff, JPhysA **41** (2008) 142001.)

Conclusions

We have considered the construction of multisoliton solution for the quadrilateral lattice equations in the ABS list.

A detailed straightforward construction exists has been done for the low members in the ABS list (H1,H2,H3,A1,Q1). (JH and Da-jun Zhang, work in progress)

Explicit NSS has also been derived for Q3, using a connection to the NQC equation. (Atkinson, JH and Nijhoff, JPhysA **41** (2008) 142001.)

The result for Q4 is less explicit. This is expected since elliptic functions enter big time. (Atkinson, JH and Nijhoff, JPhysA **40** (2007) F1.)