

# Conservation laws for difference equations

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**Observation:** Such expressions can be written in standard form,

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**Question:** Why not use equivalent expressions? This one is useful:

$$\prod_i S_i A^i(\mathbf{n}, [\mathbf{u}]) = \prod_i A^i(\mathbf{n}, [\mathbf{u}]).$$

**Answer:** The Euler operator  $\mathbf{E}$  annihilates all divergences; indeed, the variational complex is exact.

$$\dots \Lambda^{n-2,0} \xrightarrow{d_H} \Lambda^{n-1,0} \xrightarrow{d_H} \Lambda^{n,0} \xrightarrow{\mathbf{E}} \Lambda_*^{n,1} \dots$$

Divergences are the elements of  $\ker(\mathbf{E})$ .

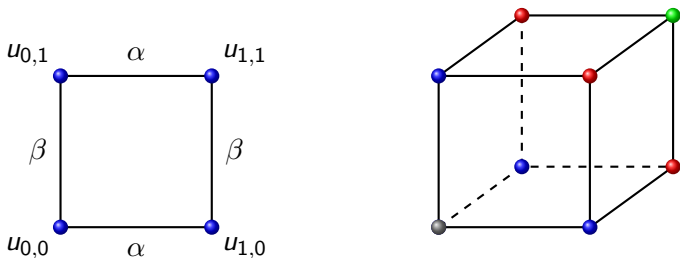
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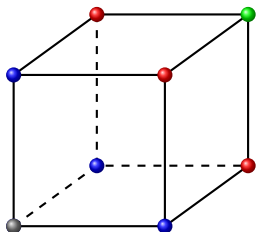
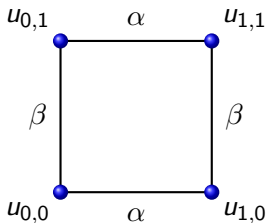
Difference operators also produce a variational complex; now  $\ker(\mathbf{E})$  is the set of all standard-form expressions.

## Quad-graph equations



Independent variables:  $k, l \in \mathbb{Z}$ ; dependent variable  $u \in \mathbb{R}$  (or  $\mathbb{C}$ ).  
 Quad-graph equations depend on  $u_{i,j} = u(k+i, l+j)$ ,  $i, j \in \{0, 1\}$ .

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Consistency-on-a-cube implies integrability (ABS classification and others);  $\alpha = \alpha(k)$ ,  $\beta = \beta(l)$ .

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ABS classification is up to Möbius transformations of  $u$  and point transformations of parameters.

$$\mathbf{Q1} : \quad \alpha(u_{0,0} - u_{0,1})(u_{1,0} - u_{1,1}) - \beta(u_{0,0} - u_{1,0})(u_{0,1} - u_{1,1}) + \delta^2 \alpha \beta (\alpha - \beta) = 0,$$

$$\mathbf{Q2} : \quad \alpha(u_{0,0} - u_{0,1})(u_{1,0} - u_{1,1}) - \beta(u_{0,0} - u_{1,0})(u_{0,1} - u_{1,1}) + \alpha \beta (\alpha - \beta)(u_{0,0} + u_{1,0} + u_{0,1} + u_{1,1}) \\ - \alpha \beta (\alpha - \beta)(\alpha^2 - \alpha \beta + \beta^2) = 0,$$

$$\mathbf{Q3} : \quad (\beta^2 - \alpha^2)(u_{0,0}u_{1,1} + u_{1,0}u_{0,1}) + \beta(\alpha^2 - 1)(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) - \alpha(\beta^2 - 1)(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) \\ - \delta^2(\alpha^2 - \beta^2)(\alpha^2 - 1)(\beta^2 - 1)/(4\alpha\beta) = 0,$$

$$\mathbf{Q4} : \quad \operatorname{sn}(\alpha)(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) - \operatorname{sn}(\beta)(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) - \operatorname{sn}(\alpha - \beta)(u_{0,0}u_{1,1} + u_{1,0}u_{0,1}) \\ + \operatorname{sn}(\alpha - \beta)\operatorname{sn}(\alpha)\operatorname{sn}(\beta)(1 + K^2 u_{0,0}u_{1,0}u_{0,1}u_{1,1}) = 0,$$

$$\mathbf{H1} : \quad (u_{0,0} - u_{1,1})(u_{1,0} - u_{0,1}) + \beta - \alpha = 0,$$

$$\mathbf{H2} : \quad (u_{0,0} - u_{1,1})(u_{1,0} - u_{0,1}) + (\beta - \alpha)(u_{0,0} + u_{1,0} + u_{0,1} + u_{1,1}) + \beta^2 - \alpha^2 = 0,$$

$$\mathbf{H3} : \quad \alpha(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) - \beta(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) + \delta^2(\alpha^2 - \beta^2) = 0,$$

$$\mathbf{A1} : \quad \alpha(u_{0,0} + u_{0,1})(u_{1,0} + u_{1,1}) - \beta(u_{0,0} + u_{1,0})(u_{0,1} + u_{1,1}) - \delta^2 \alpha \beta (\alpha - \beta) = 0,$$

$$\mathbf{A2} : \quad (\beta^2 - \alpha^2)(u_{0,0}u_{1,0}u_{0,1}u_{1,1} + 1) + \beta(\alpha^2 - 1)(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) - \alpha(\beta^2 - 1)(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) = 0.$$

## Three-point conservation laws

**Definition:** A *conservation law* (CLaw) of a quad-graph equation is an expression

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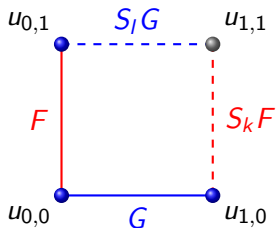
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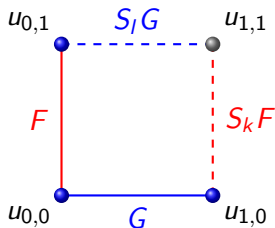
**Problem:** Find a basis for the vector space of all equivalence classes of CLaws.

Too hard — start small.



The simplest nontrivial CLaws have

$$F = F(k, l, u_{0,0}, u_{0,1}), \quad G = G(k, l, u_{0,0}, u_{1,0}).$$



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The determining equation for  $u_{1,1} = \omega$  is

$$F(k+1, l, u_{1,0}, \omega) - F(k, l, u_{0,0}, u_{0,1}) + G(k, l+1, u_{0,1}, \omega) - G(k, l, u_{0,0}, u_{1,0}) = 0.$$

## Solution Method

Apply the commuting operators

$$L_1 = \frac{\partial}{\partial u_{1,0}} - \frac{\omega_{1,0}}{\omega_{0,0}} \frac{\partial}{\partial u_{0,0}}, \quad L_2 = \frac{\partial}{\partial u_{0,1}} - \frac{\omega_{0,1}}{\omega_{0,0}} \frac{\partial}{\partial u_{0,0}},$$

to get

$$-L_1 L_2 \{ F(k, l, u_{0,0}, u_{0,1}) + G(k, l, u_{0,0}, u_{1,0}) \} = 0.$$

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Elimination can be made faster for some quad-graph equations.

**Example:** For the dpKdV equation **(H1)**

$$u_{1,1} = u_{0,0} + \frac{\beta - \alpha}{u_{1,0} - u_{0,1}},$$

the set of three-point CLaws (up to equivalence) is spanned by

$$F_1 = -(-1)^{k+l} (2u_{0,0}u_{0,1} - \beta), \quad G_1 = (-1)^{k+l} (2u_{0,0}u_{1,0} - \alpha),$$

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Other ABS equations:

4 CLaws : **Q1**<sub>δ=0</sub>(cross-ratio), **A1**<sub>δ=0</sub>, **H3**<sub>δ=0</sub>(dpmKdV);

1 CLaw : **Q2**, **Q3**<sub>δ=1</sub>, **Q4**<sub>K<sup>2</sup>≠1</sub>;

2 CLaws : all other equations.

The same process can be used to find higher conservation laws for

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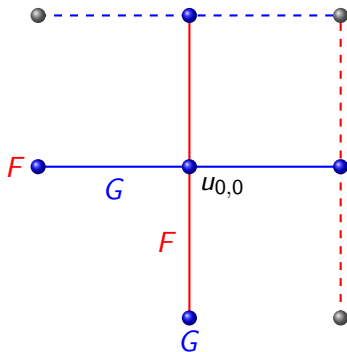
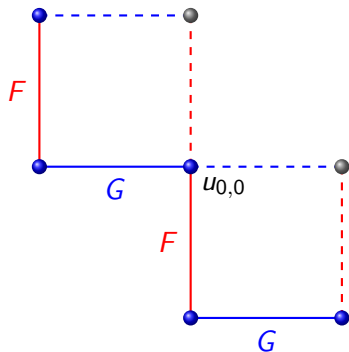
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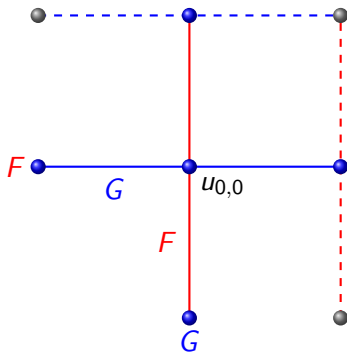
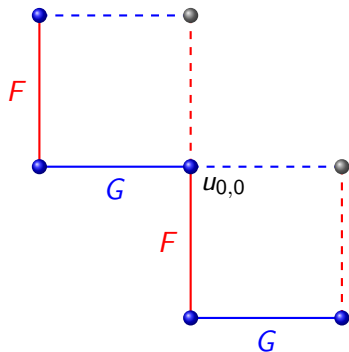
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For ABS, the cross is more efficient.

**Example:** The dpKdV equation has the following five-point CLaws (modulo three-point and trivial CLaws):

$$F_1 = -\ln(u_{0,1} - u_{-1,0}), \quad G_1 = \ln(u_{1,0} - u_{-1,0}),$$

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For constant  $\alpha, \beta$ :

**every ABS equation has something similar** (at least).

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to get (up to equivalence) the new CLaw

$$F = \frac{-1}{(u_{0,0} - u_{-2,0})(u_{0,1} - u_{-1,0})}, \quad G = \frac{1}{(u_{0,0} - u_{-2,0})(u_{1,0} - u_{-1,0})}.$$

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**Warning:** apparent novelty must be checked; use characteristics.

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**Drawback:** The resulting CLaws found so far are quite messy!

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However, there are no five-point symmetries or CLaws if neither  $\alpha$  nor  $\beta$  are constant.

## Open Problems

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- Does each of the ABS equations have infinitely many CLaws (up to equivalence)? If so, is this a property shared by all integrable quad graphs?
- Is there a one-to-one correspondence between variational symmetries and CLaws (up to equivalence)?

**The End**