

Solutions to 2×2 iso-monodromy problems for the second and third q -discrete Painlevé equations

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Examples of some Painlevé equations

- 2nd order nonlinear integrable equations

$$\text{PII} \quad \frac{d^2 y}{dz^2} = 2y^3 + zy + \alpha$$

$$\text{PIII} \quad \frac{d^2 y}{dz^2} = \frac{1}{y} \left(\frac{dy}{dz} \right)^2 - \frac{1}{z} \frac{dy}{dz} + \frac{\alpha y^2 + \beta}{z} + \gamma y^3 + \frac{\delta}{y}$$

- Iso-monodromy deformation (Lax Pair)
- Symmetry reduction from well known partial differential equation
- Transcendental general solutions
- Special solutions
- Backlund transformation

q-discrete Painlevé equations

- Second order q-discrete nonlinear equations

$$\text{q-PII} \quad g(q^{-1}z)g(qz) = \frac{-\gamma_0 g(z)z + \alpha_0 z^2}{g(z)(g(z) - \gamma_0)}$$

$$\text{q-PIII} \quad g(qz)g(q^{-1}z) = \frac{cd(g(z) - az)(g(z) - bz)}{(g(z) - c)(g(z) - d)}$$

- Resulting from singularity confinement
- Symmetry reduction from well known q-partial discrete equations
- Iso-monodromy deformation (Lax Pair)
- Transcendental general solutions
- Special solutions
- Backlund transformation

Iso-monodromy problem for q-PIII

- Spectral Problem (monodromy data is computed)

$$\Psi(qx) = A(x, z)\Psi(x)$$

$$A(x, z) = \begin{pmatrix} 0 & a_1(z) \\ a_2(z) & 0 \end{pmatrix} x^2 + \begin{pmatrix} m_1(z) & 0 \\ 0 & m_2(z) \end{pmatrix} x + \begin{pmatrix} 0 & b_1(z) \\ b_2(z) & 0 \end{pmatrix}$$

- Deformation

$$\begin{aligned} \Psi(x, qz) &= B(x, z)\Psi(x, z) \\ &= \begin{pmatrix} \frac{g(qz)}{\sqrt{qg(z)}} & -\frac{z^{3/2}\delta}{\sqrt{qg(z)}}x \\ -\sqrt{z}\delta g(qz)x & 1 \end{pmatrix} \Psi(x, z) \end{aligned}$$

- Compatibility Condition: $A(x, qz)B(x, z) = B(qx, z)A(x, z)$

$$A(x, qz)B(x, z) = B(qx, z)A(x, z) \Rightarrow$$

$$g(qz)g(q^{-1}z) = \frac{\alpha_0 g(z)^2 + \frac{\gamma_0}{q^2} z^2}{g(z)^2 + \alpha_0}$$

- This is a special case of the full q-PIII

$$g(qz)g(q^{-1}z) = \frac{cd(g(z) - az)(g(z) - bz)}{(g(z) - c)(g(z) - d)}$$

with

$$a = -b$$

$$c = -d$$

Why study the iso-monodromy problem of q-Painlevé equations?

- Associated linear problem for q-PIII

$$\begin{aligned}\Psi(qx) &= A(x, z)\Psi(x) \\ &= (A_0(z) + A_1(z)x + A_2(z)x^2) \Psi(x)\end{aligned}$$

- Inverse iso-monodromy deformation method for differential system (Flaschka and Newell (1980))
- Availability of simple 2×2 Lax pairs (M.Hay (2007))
- General theory for q-discrete Linear system
- q-discrete Special functions
- Some known special solutions to q-Painlevé equations

Birkhoff's formula (1913)

q-Discrete Linear System

$$\Psi(qx) = A(x)\Psi(x)$$

$$A(x) = A_0 + A_1x + A_2x^2 + \dots + A_\mu x^\mu$$

where Ψ , A_i are $n \times n$ matrices.

Systems of solutions

$$\begin{cases} \Psi_0(x) = (x^{\rho_j} a_{ij}(x)) \\ \Psi_\infty(x) = q^{\frac{\mu}{2}(t^2-t)} (x^{-\sigma_j} b_{ij}(x)) \\ \Psi_0(x) = \Psi_\infty(x)P(x) \end{cases}$$

q^{ρ_j} , the n eigenvalues of A_0

$q^{-\sigma_j}$, the n eigenvalues of A_μ

$$t = \frac{\ln x}{\ln q}$$

Two cases of when a q-linear system can be explicitly solved in terms of elementary known functions

$$\Psi(qx) = (A_0 + A_1x + A_2x^2 + \dots + A_\mu x^\mu)\Psi(x)$$

where Ψ , A_i are $n \times n$ matrices.

$$n = 1, \mu = 1 \quad \psi(qx) = (1 - x)\psi(x)$$

- solved by q-Gamma function

$$n = 2, \mu = 1 \quad \Psi(qx) = (A_0 + A_1x)\Psi(x)$$

- solved by q-Hypergeometric function

Associated Linear System for q-Painlevé equations $n = 2, \mu > 1$

- $\Psi(qx) = (A_0 + A_1x + A_2x^2)\Psi(x)$ for qPIII
- $\Psi(qx) = (A_0 + A_1x + A_2x^2 + A_3x^3)\Psi(x)$ for qPII

When does linear system simplifies?

$$\text{q-PIII:} \quad g(q^2z)g(z) = \frac{\alpha_0 g(qz)^2 + \gamma_0 z^2}{g(qz)^2 + \alpha_0}$$

$$\begin{aligned} \Psi(qx) &= (A_2(z)x^2 + A_1(z)x + A_0(z))\Psi(x) \\ &= \left\{ \begin{pmatrix} 0 & a_1(z) \\ a_2(z) & 0 \end{pmatrix} x^2 + \begin{pmatrix} m_1(z) & 0 \\ 0 & m_2(z) \end{pmatrix} x + \begin{pmatrix} 0 & b_1(z) \\ b_2(z) & 0 \end{pmatrix} \right\} \Psi(x) \end{aligned}$$

- $A_0(z), A_1(z), A_2(z)$ commute $\Rightarrow g(z) = \left(\frac{\gamma_0}{q^2}\right)^{1/4} \sqrt{z}$

Linear system simplifies in case of q-PIII having algebraic special solutions $g(z) = \left(\frac{\gamma_0}{q^2}\right)^{1/4} \sqrt{z}$

let $\Psi = C\Psi_1$,

$$\begin{aligned}\Psi_1(qx) &= C^{-1}A(x, z)C\Psi_1(x) \\ &= (C^{-1}A_2Cx^2 + C^{-1}A_1Cx + C^{-1}A_0C)\Psi_1\end{aligned}$$

$$\begin{aligned}&\begin{pmatrix} u(qx) \\ v(qx) \end{pmatrix} \\ &= \begin{pmatrix} \gamma_0^{1/4} a_2(z)x^2 + m_1(z)x + \gamma_0^{1/4} b_2(z) & 0 \\ 0 & -\gamma_0^{1/4} a_2(z)x^2 + m_1(z)x - \gamma_0^{1/4} b_2(z) \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}\end{aligned}$$

Linear system simplifies in case of q-PIII having algebraic special solutions $g(z) = \left(\frac{\gamma_0}{q^2}\right)^{1/4} \sqrt{z}$

$$\begin{aligned}u(qx) &= (\gamma_0^{1/4} a_2(z)x^2 + m_1(z)x + \gamma_0^{1/4} b_2(z))u(x) \\ &= c_1\left(x - \frac{q}{z}\right)\left(x - \frac{\gamma_0}{\alpha_0}\right)u(x)\end{aligned}$$

- Let $u = c_1^t u_1 u_2$, $t = \frac{\ln x}{\ln q}$

$$\begin{aligned}u(qx) &= c_1^{t+1} u_1(qx) u_2(qx) \\ &= \left(c_1\left(x - \frac{q}{z}\right)\left(x - \frac{\gamma_0}{\alpha_0}\right)\right) c_1^t u_1(x) u_2(x)\end{aligned}$$

- $u_1(qx) = \left(x - \frac{q}{z}\right)u_1(x)$, $u_2(qx) = \left(x - \frac{\gamma_0}{\alpha_0}\right)u_2(x)$

$$\begin{aligned}
 \Psi(qx, z) &= A(x, z)\Psi(x, z) \\
 &= \left\{ \begin{pmatrix} e_1(z) & 0 \\ 0 & e_2(z) \end{pmatrix} x^3 + \begin{pmatrix} 0 & m_1(z) \\ m_2(z) & 0 \end{pmatrix} x^2 \right. \\
 &\quad \left. + \begin{pmatrix} n_1(z) & 0 \\ 0 & n_2(z) \end{pmatrix} x + \begin{pmatrix} 0 & b_1(z) \\ b_2(z) & 0 \end{pmatrix} \right\} \Psi(qx, z)
 \end{aligned}$$

$$\begin{aligned}
 \Psi(x, qz) &= B(x, z)\Psi(x, z) \\
 &= \begin{pmatrix} q^{-1/2}g(z/q)\sqrt{z} & -\frac{\sqrt{z}g(z/q)g(z)}{q^{-1/2}} \\ -\frac{z^{-3/2}}{xg(qz)g(z)} & \frac{q^{-1/2}}{\sqrt{z}g(z/q)} \end{pmatrix} \Psi(x, z)
 \end{aligned}$$

Associated linear system for q-PII

q-PII

$$g(q^{-1}z)g(qz) = \frac{-\gamma_0 g(z)z + \alpha_0 z^2}{g(z)(g(z) - \gamma_0)}$$

Associated linear system

$$\begin{aligned}\Psi(qx) &= A(x, z)\Psi(x) \\ &= \left\{ \begin{pmatrix} e_1(z) & 0 \\ 0 & e_2(z) \end{pmatrix} x^3 + \begin{pmatrix} 0 & m_1(z) \\ m_2(z) & 0 \end{pmatrix} x^2 \right. \\ &\quad \left. + \begin{pmatrix} n_1(z) & 0 \\ 0 & n_2(z) \end{pmatrix} x + \begin{pmatrix} 0 & f_1(z) \\ f_2(z) & 0 \end{pmatrix} \right\} \Psi(qx)\end{aligned}$$

$$q\text{-PII} : g(q^{-1}z)g(qz) = \frac{-\gamma_0 g(z)z + \alpha_0 z^2}{g(z)(g(z) - \gamma_0)}$$

Asymptotic expansions as $x \rightarrow \infty$

$$\Psi(x, z) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} a_1(z) & b_1(z) \\ c_1(z) & d_1(z) \end{pmatrix} + \frac{1}{x^2} \begin{pmatrix} a_2(z) & b_2(z) \\ c_2(z) & d_2(z) \end{pmatrix} + \dots \right\} \\ \times q^{\frac{3}{2}(t^2 - t)} \begin{pmatrix} e_1^t & 0 \\ 0 & e_2^t \end{pmatrix}$$

$$b_n(e_2/q^n - e_1) = F(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1})$$

$$c_n(e_1/q^n - e_2) = F(a_{n-1}, b_{n-1}, c_{n-1}, d_{n-1})$$

$$e_2 = q^n e_1 \Rightarrow \alpha_0 = \frac{1}{q^n}$$

$$e_1 = q^n e_2 \Rightarrow \alpha_0 = q^n$$

for n odd

$$q\text{-PII} : g(q^{-1}z)g(qz) = \frac{-\gamma_0 g(z)z + \alpha_0 z^2}{g(z)(g(z) - \gamma_0)}$$

- Two simple examples:

$$e_2 = qe_1 \Rightarrow \alpha_0 = 1/q$$

$$e_2 = q^{-1}e_1 \Rightarrow \alpha_0 = q$$

$$\Psi(x, z) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} 0 & b_1(z) \\ c_1(z) & 0 \end{pmatrix} + \frac{1}{x^2} \begin{pmatrix} a_2(z) & 0 \\ 0 & d_2(z) \end{pmatrix} + \dots \right\} \\ \times q^{\frac{3}{2}(t^2-t)} \begin{pmatrix} e_1^t & 0 \\ 0 & e_2^t \end{pmatrix}$$

- Equating coefficients of the same order in x we get conditions on $b_1(z)$, $c_1(z)$

$$b_1(z)(e_2/q - e_1) = m_1(z) \quad \text{if} \quad \alpha_0 = \frac{1}{q} \quad \Rightarrow \quad g(qz) = \frac{-z}{g(z) - \gamma_0}$$

$$c_1(z)(e_1/q - e_2) = m_2(z) \quad \text{if} \quad \alpha_0 = q \quad \Rightarrow \quad g(qz) = \frac{\gamma_0 - qz}{g(z)}$$

Special Solutions for q-PII

- q-PII

$$g(q^{-1}z)g(qz) = \frac{-\gamma_0 g(z)z + \alpha_0 z^2}{g(z)(g(z) - \gamma_0)}$$

Riccati special solution

$$g(qz) = \frac{-z}{g(z) - \gamma_0}, \quad \alpha_0 = \frac{1}{q}$$

- Linearise \Rightarrow q-Airy equation

$$g(z) = \frac{u(z)}{v(z)} \quad u(qz) = -zv(z)$$

$$v(q^2z) + \gamma_0 v(qz) + zv(z) = 0$$

- Follow the analogous behaviors of the differential and q -discrete systems, especially the iso-monodromy problems for q -PII and q -PIII
- Analyze the spectral half the iso-monodromy problems in the frame work of q -discrete linear systems developed by Birkhoff and Adam et al.
- Special cases which the associated linear system can be simplified correspond to q -Painlevé equation having special solutions
- The q -discrete Lax pair contains all the information and properties of its associated q -Painlevé equation, hence will provide a way of studying these very special q -nonlinear equations.

THANK YOU!