

Symmetries and reductions of integrable discrete equations on quad-graphs

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Outline

- Integrable equations on quad-graphs
Their key properties
- Applications of symmetry methods
- Geometrical (Lie-point) symmetries and Yang-Baxter maps
- Generalized symmetries and reductions to discrete Painlevé equations

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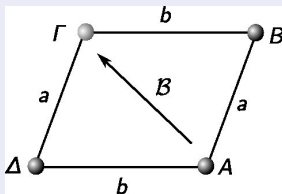
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Integrable equations on quad-graphs

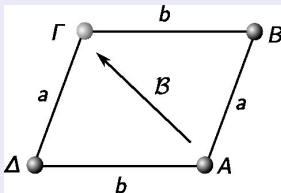


- $Q(f_A, f_B, f_\Gamma, f_\Delta; a, b) = 0$
- Affine linearity
- Symmetries of the square
- $f_\Gamma = \varphi(f_A, f_B, f_\Delta; a, b)$.

$$\mathcal{B}_{(a,b)} : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{X} \times \mathbb{X} \times \mathbb{X}$$

$$\mathcal{B}_{(a,b)}(f_\Delta, f_A, f_B) = (f_\Delta, f_\Gamma, f_B),$$

Integrable equations on quad-graphs

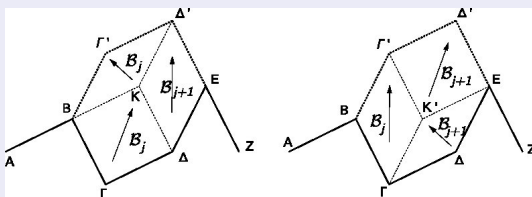


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Integrability and the braid relation



$$\mathcal{B}_j : \mathbb{X}^n \rightarrow \mathbb{X}^n \quad \mathcal{B}_j = \text{Id}_{\mathbb{X}} \times \cdots \times \mathcal{B}_{(a_{j-1}, a_j)} \times \cdots \times \text{Id}_{\mathbb{X}},$$

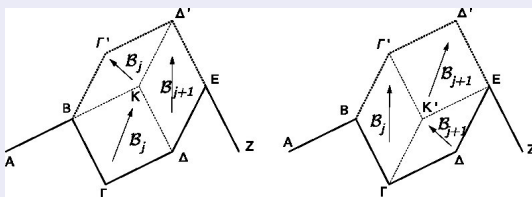
where \mathcal{B} acts on the $j - 1$, j and the $j + 1$ factors of the map.

- Key properties of maps associated to integrable discrete equations on quad-graphs are the relations

$$\mathcal{B}_j^2 = \text{Id}, \quad (\mathcal{B}_j \mathcal{B}_{j+1})^3 = \text{Id}, \quad \mathcal{B}_j \mathcal{B}_i = \mathcal{B}_i \mathcal{B}_j \quad |i - j| > 1$$

- $\mathcal{B}_j \mathcal{B}_{j+1} \mathcal{B}_j = \mathcal{B}_{j+1} \mathcal{B}_j \mathcal{B}_{j+1}$

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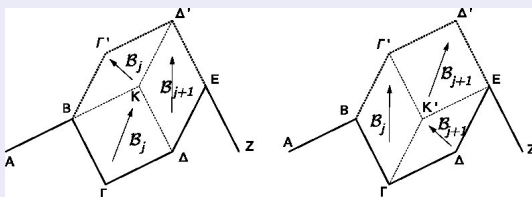
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Lie point symmetries

We consider one-parameter group of transformations

$$G : (x, y, z) \mapsto (X(x; \varepsilon), Y(y; \varepsilon), Z(z; \varepsilon)) ,$$

If G commutes with $\mathcal{B}_{(a,b)}$ i.e. $G \mathcal{B}_{(a,b)} = \mathcal{B}_{(a,b)} G$, it is a symmetry of the map $\mathcal{B}_{(a,b)}$ and consequently of the equation $Q = 0$

The map associated with the discrete KdV equation

$$\mathcal{B}_{(a,b)}(x, y, z) = \left(x, y + \frac{a - b}{x - z}, z\right),$$

- The group of Lie-point symmetries of dKdV equation

$$G_1 : (x, y, z) \mapsto (x + \varepsilon_1, y + \varepsilon_1, z + \varepsilon_1) ,$$

$$G_2 : (x, y, z) \mapsto (x - \varepsilon_2, y + \varepsilon_2, z - \varepsilon_2) , \quad SO(1, 1)$$

$$G_3 : (x, y, z) \mapsto (x e^{-\varepsilon_3}, y e^{\varepsilon_3}, z e^{-\varepsilon_3}) ,$$

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Projection of the map $B_{(a,b)}$ to \mathbb{X}^3/G

$$B_{(a,b)}(x, y, z) = \left(x, y + \frac{a-b}{x-z}, z\right)$$

$$G_1 : (x, y, z) \mapsto (x + \varepsilon_1, y + \varepsilon_1, z + \varepsilon_1),$$

The action of G_1 on \mathbb{X}^3 is regular with one-dimensional orbits.

Local coordinates on the two-dimensional quotient space \mathbb{X}^3/G_1 :

$$u = y - x, \quad v = z - y.$$

Projection of the map to \mathbb{X}^3/G_1 gives $B_{(a,b)} : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{X} \times \mathbb{X}$

$$B_{(a,b)}(u, v) = \left(u - \frac{a-b}{u+v}, v + \frac{a-b}{u+v}\right),$$

Braid equation

$$(\text{Id}_{\mathbb{X}} \times B)(B \times \text{Id}_{\mathbb{X}})(\text{Id}_{\mathbb{X}} \times B) = (B \times \text{Id}_{\mathbb{X}})(\text{Id}_{\mathbb{X}} \times B)(B \times \text{Id}_{\mathbb{X}}).$$

Yang-Baxter maps

$$R_{(a,b)} = \sigma B_{(a,b)}$$

where σ is the twist map

$$R_{(a,b)}(u, v) = \left(v + \frac{a-b}{u+v}, u - \frac{a-b}{u+v} \right),$$

is a YB map known as the Adler map

- Generalizations:

The method can be applied to multi-field equations by using the multi-parameter groups of symmetry transformations and the multi-dimensional braid relations

V.Papageorgiou, AT and A.Veselov (2006),
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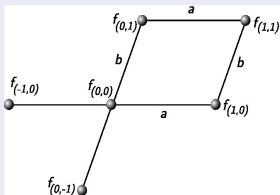
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Generalized symmetries of quadrilateral equations



- $Q(f_{(0,0)}, f_{(1,0)}, f_{(0,1)}, f_{(1,1)}; a, b) = 0$
- Affine linearity, D_4 symmetry

Infinitesimal generators of Lie-point, three-point and five-point generalized symmetries

$$\mathbf{v} = R(n, m, f_{(0,0)}) \partial_{f_{(0,0)}}$$

$$\mathbf{v} = R(n, m, f_{(0,0)}, f_{(1,0)}, f_{(-1,0)}; a, b) \partial_{f_{(0,0)}}$$

$$\mathbf{v} = R(n, m, f_{(0,0)}, f_{(1,0)}, f_{(-1,0)}, f_{(0,1)}, f_{(0,-1)}; a, b) \partial_{f_{(0,0)}}$$

Three point generalized symmetries

Infinitesimal invariance condition:

$$\mathbf{v} Q = 0 \quad \text{mod } Q = 0$$

Biquadratic polynomials

$$h(f_{(0,0)}, f_{(1,0)}; a, b) = Q_{Q, f_{(0,1)} f_{(1,1)}} - Q_{Q, f_{(0,1)}} Q_{Q, f_{(1,1)}}$$

$$G(f_{(0,0)}, f_{(1,1)}; a, b) = Q_{Q, f_{(1,0)} f_{(0,1)}} - Q_{Q, f_{(1,0)}} Q_{Q, f_{(0,1)}}$$

Theorem: Every two-dimensional lattice equation $Q = 0$, where the function Q is affine linear and D_4 -symmetric, admits a three-point generalized symmetry with generator

$$\mathbf{v}_n = \left(\frac{h(f_{(0,0)}, f_{(1,0)}; a, b)}{f_{(1,0)} - f_{(-1,0)}} - \frac{1}{2} h_{f_{(1,0)}}(f_{(0,0)}, f_{(1,0)}; a, b) \right) \partial_{f_{(0,0)}}.$$

Three point generalized symmetries

In the generic case, where the matrix

$$\mathcal{G} = \left(\begin{array}{ccc} h(x, y) & G(x, z) & G(x, w) \\ h_{,x}(x, y) & G_{,x}(x, z) & G_{,x}(x, w) \\ h_{,xx}(x, y) & G_{,xx}(x, z) & G_{,xx}(x, w) \end{array} \right) \Big|_{x=0}$$

has rank 3, every three-point generalized symmetry generator necessarily has the form

$$\mathbf{V}_n = A(n; a, b) \mathbf{v}_n + \frac{1}{2} \phi(n, m, f_{(0,0)}; a, b) \partial_{f_{(0,0)}}$$

$$(A(n; a, b) - A(n+1; a, b)) h(f_{(0,0)}, f_{(1,0)})^2 \partial_{f_{(1,0)}} \left(\frac{G(f_{(1,0)}, f_{(0,1)})}{h(f_{(0,0)}, f_{(1,0)})} \right)$$

$$+ G(f_{(1,0)}, f_{(0,1)}) \phi(n, m, f_{(0,0)}; a, b) + h(f_{(0,0)}, f_{(0,1)}) \phi(n+1, m, f_{(1,0)}; a, b)$$

$$+ h(f_{(0,0)}, f_{(1,0)}) \phi(n, m+1, f_{(0,1)}; a, b) = Q_{,f_{(1,1)}}^2 \phi(n+1, m+1, f_{(1,1)}; a, b).$$

Five point generalized symmetries - the generic case

Theorem: If $\text{rank } \mathcal{G} = 3$, Q is affine linear and D_4 symmetric then the characteristic R of a five-point symmetry generator of $Q = 0$ has the form

$$R = A(n; a, b)P(f_{(0,0)}, f_{(1,0)}, f_{(-1,0)}; a, b) \\ + B(m; a, b)P(f_{(0,0)}, f_{(0,1)}, f_{(0,-1)}; b, a) + \frac{\psi(n, m, f_{(0,0)}; a, b)}{2}$$

where

$$P(u, x, y; a, b) = \frac{h(u, x; a, b)}{x - y} - \frac{1}{2}h_{,x}(u, x; a, b),$$

The determining equation

$$\begin{aligned} & (A(n; a, b) - A(n + 1; a, b)) h(f_{(0,0)}, f_{(1,0)})^2 \partial_{f_{(1,0)}} \left(\frac{G(f_{(1,0)}, f_{(0,1)})}{h(f_{(0,0)}, f_{(1,0)})} \right) \\ & + (B(m; a, b) - B(m + 1; a, b)) h(f_{(0,0)}, f_{(0,1)})^2 \partial_{f_{(0,1)}} \left(\frac{G(f_{(1,0)}, f_{(0,1)})}{h(f_{(0,0)}, f_{(0,1)})} \right) \\ & + G(f_{(1,0)}, f_{(0,1)}) \psi(n, m, f_{(0,0)}; \alpha, \beta) + h(f_{(0,0)}, f_{(0,1)}) \psi(n + 1, m, f_{(1,0)}; a, b) \\ & + h(f_{(0,0)}, f_{(1,0)}) \psi(n, m + 1, f_{(0,1)}; \alpha, \beta) = Q_{f_{(1,1)}}^2 \psi(n + 1, m + 1, f_{(1,1)}; a, b), \end{aligned}$$

- Exhaustive list of the corresponding Lie point, three- and five-point generalized symmetry generators of the Adler-Bobenko-Suris classified integrable discrete equations

AT, D. Tsubelis and P. Xenitidis (2007)

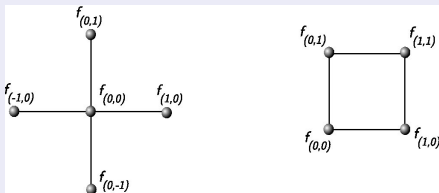
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Invariant solutions

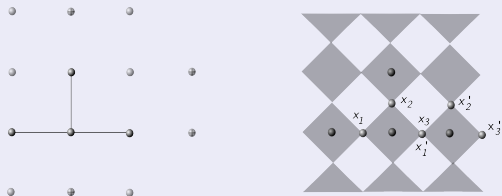


A function $f : \mathbb{Z}^2 \mapsto \mathbb{C}$ is called an invariant solution of the lattice equation $Q = 0$ under the symmetry $\mathbf{v} = R[f]\partial_{f_{(0,0)}}$, if it satisfies the lattice equation $Q = 0$ and the compatible constraint $\mathbf{v}(f) = 0$.

$$R[f] = 0$$

$$Q[f] = 0$$

Symmetry reductions and Cauchy problems



$$\mathbf{L} = R_L(n, m, f_{(0,0)}) \partial_{f_{(0,0)}}$$

$$\mathbf{v}_C = \mathbf{v}_n + \lambda \mathbf{v}_m$$

$$\mathcal{C}[f] := R(f_{(0,0)}, f_{(1,0)}, f_{(-1,0)}; a, b) + \lambda R(f_{(0,0)}, f_{(0,1)}, f_{(0,-1)}; b, a) = 0,$$

where

$$R(u, x, y; \alpha, \beta) = \frac{h(u, x; a, b)}{x - y} - \frac{1}{2} h_{,x}(u, x; a, b).$$

$$[\mathbf{L}, \mathbf{v}_C] = 0, \quad \text{mod } \mathcal{C}[f] = 0$$

$$P : (x_1, x_2, x_3) \rightarrow (x'_1, x'_2, x'_3)$$

Symmetry reduction of discrete KdV

$$\mathbf{dKdV} \quad (f_{(0,0)} - f_{(1,1)}) (f_{(1,0)} - f_{(0,1)}) - a + b = 0$$

Point symmetries :

$$\mathbf{L}_1 = \partial_{u_{(0,0)}}, \quad \mathbf{L}_2 = (-1)^{n+m} \partial_{u_{(0,0)}}, \quad \mathbf{L}_3 = (-1)^{n+m} u_{(0,0)} \partial_{u_{(0,0)}}$$

Three-point generalized symmetries :

$$\mathbf{v}_1 = \frac{1}{f_{(1,0)} - f_{(-1,0)}} \partial_{f_{(0,0)}}, \quad \mathbf{v}_2 = n \mathbf{v}_1 + \frac{f_{(0,0)}}{2(a-b)} \partial_{f_{(0,0)}},$$

$$\mathbf{v}_3 = \frac{1}{f_{(0,1)} - f_{(0,-1)}} \partial_{f_{(0,0)}}, \quad \mathbf{v}_4 = m \mathbf{v}_3 - \frac{f_{(0,0)}}{2(a-b)} \partial_{f_{(0,0)}}.$$

$$f \mapsto (-1)^{n+m} f, \quad \mathbf{L}_2 \mapsto \mathbf{L}_1$$

$$\mathbf{v}_C = \sum_{i=1}^3 \lambda_i \mathbf{L}_i + \sum_{i=1}^4 \mu_i \mathbf{v}_i$$

Symmetry reduction of discrete KdV using L_1

$$v_{(0,0)} = f_{(1,0)} - f_{(0,0)}, \quad w_{(0,0)} = f_{(1,1)} - f_{(1,0)},$$

$$v_{(0,1)} = f_{(1,1)} - f_{(0,1)}, \quad w_{(-1,0)} = f_{(0,1)} - f_{(0,0)}$$

$$[\mathbf{L}_1, \mathbf{v}_C] = \lambda_3 \mathbf{L}_2 + \frac{1}{2} \frac{\mu_2 - \mu_4}{a - b} \mathbf{L}_1$$

$$\lambda_3 = 0, \quad \mu_2 = \mu_4.$$

$$C[g] := \lambda_1 + \lambda_2 (-1)^{n+m} + \frac{\mu_2 n + \mu_1}{g_1 + g_2} + \frac{\mu_2 m + \mu_3}{g_3 + g_4} = 0.$$

where

$$(g_1, g_2, g_3, g_4) = (v_{(0,0)}, v_{(-1,0)}, w_{(-1,0)}, w_{(-1,-1)}).$$

The asymmetric, alternate discrete Painlevé II equation

$$x_1' = x_3,$$

$$x_2' = -x_3 + \frac{r}{x_3 - x_2},$$

$$x_3' = -x_3 + \frac{r c(n+1, \mu_1)}{\left(\frac{c(n, \mu_1)}{x_1 + x_3} + \lambda(n+m)\right) (x_2 - x_3)^2 + c(m, \mu_3) (x_2 - x_3) - r \lambda(n+m+1)},$$

where $c(n, \mu) = \mu_2 n + \mu$, $\lambda(n) = \lambda_1 + \lambda_2 (-1)^n$ and $r = a - b$.

The above system can be decoupled for the variable

$$y(n) := x_2' + x_3 = x_2(n+1, m) + x_3(n, m)$$

$$\frac{r c(n+1, \mu_1)}{y(n+1)y(n) + r} + \frac{r c(n, \mu_1)}{y(n)y(n-1) + r} = c(n+1, \mu_1) + c(m, \mu_3) + y(n) \lambda(n+m+1) - \frac{r}{y(n)} \lambda(n+m)$$

Future work

- Complete lists of group invariant solutions of the integrable discrete equations in the Adler-Bobenko-Suris classification - new versions of discrete Painlevé equations
- Symmetries of integrable discrete equations without the D_4 symmetry - mappings arising from the corresponding symmetry reductions
- Generalizations to higher dimensions (in both dependent and independent variables)