A Extended Two Dimensional Toda Lattice Hierarchy and Its Solutions

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Introduction

Two dimensional Toda lattice hierarchy (2dTLH) (Ueno & Takasaki, 1982 [1])

- includes two dimensional Toda lattice (2dTL)

$$\partial_x \partial_y q(n) = e^{q(n)-q(n-1)} - e^{q(n+1)-q(n)}$$

- differential operator (∂) $\longrightarrow$ *difference operator (Λ)*

multi-component generalization (U-T, [1])

- scalar-difference operator $\longrightarrow$ *matrix-difference operator*
- include non-abelian Toda hierarchy

Introduction

We present a new extended two-dimensional Toda lattice hierarchy (ex2dTLH)

- introduce new vector field for 2dTLH
- include two dimensional Toda lattice with self-consistent sources (2dTLSCS, Hu X.B, Wang H.Y. 2006 [2])

We solve 2dTLSCS:

- construct Darboux transformation (DT) for 2dTLSCS
- apply method of variation of constants (MVC) to give non-auto-Bäcklund DT ($N \to N + 1$ SCSs)

Two dimensional Toda Lattice Hierarchy

2dTLH [1]

Let $\mathbf{x} = (x_1, x_2, \cdots)$, $\mathbf{y} = (y_1, y_2, \cdots)$, $n \in \mathbb{Z}$. Let

$$L = \Lambda + \sum_{i \geq 0} u_i \Lambda^{-i} \qquad M = \sum_{i \geq -1} v_i \Lambda^i$$

Λ is *shift operator* such that $\Lambda f(n) = f(n+1)\Lambda$. Then 2dTLH

$$\frac{\partial}{\partial x_m} L = [B_m, L] \qquad \frac{\partial}{\partial y_m} L = [C_m, L]$$
$$\frac{\partial}{\partial x_m} M = [B_m, M] \qquad \frac{\partial}{\partial y_m} M = [C_m, M]$$

$B_m = (L^m)_{\geq 0}$, $C_m = (M^m)_{<0}$

Two dimensional Toda Lattice Hierarchy

Equivalent zero-curvature form

$$B_{k,x_m} - B_{m,x_k} + [B_k, B_m] = 0 \tag{2a}$$
$$C_{k,y_m} - C_{m,y_k} + [C_k, C_m] = 0 \tag{2b}$$
$$B_{k,y_m} - C_{m,x_k} + [B_k, C_m] = 0 \tag{2c}$$

Example

$m = k = 1,\ v_{-1} = e^{q(n)-q(n-1)},\ u_0 = q_x \Rightarrow$ 2dTL

$$\partial_{x_1}\partial_{y_1} q(n) = e^{q(n)-q(n-1)} - e^{q(n+1)-q(n)} \qquad n \in \mathbb{Z}$$

Two dimensional Toda Lattice Hierarchy

Wave operators

$$\hat{W}^{(\infty)} = 1 + b_1\Lambda^{-1} + b_2\Lambda^{-2} + \cdots$$
$$\hat{W}^{(0)} = c_0 + c_1\Lambda + c_2\Lambda^2 + \cdots$$

Proposition (U-T)

If L, M satisfy 2dTLH (1), then exists wave operators such that

$$L = \hat{W}^{(\infty)}\Lambda\hat{W}^{(\infty)-1} \qquad M = \hat{W}^{(0)}\Lambda^{-1}\hat{W}^{(0)-1}$$
$$\partial_{x_m}\hat{W}^{(\infty)} = -L^m_{<0}\hat{W}^{(\infty)} \qquad \partial_{x_m}\hat{W}^{(0)} = L^m_{\geq 0}\hat{W}^{(0)}$$
$$\partial_{y_m}\hat{W}^{(\infty)} = M^m_{<0}\hat{W}^{(\infty)} \qquad \partial_{y_m}\hat{W}^{(0)} = -M^m_{\geq 0}\hat{W}^{(0)}$$

Two dimensional Toda Lattice Hierarchy

Wave function & adjoint wave function

$$w^{(\infty)} = \hat{W}^{(\infty)}(\lambda^n)e^{\xi(\mathbf{x},\lambda)} \qquad w^{(\infty)*} = \hat{W}^{(\infty)*-1}(\lambda^{-n})e^{-\xi(\mathbf{x},\lambda)}$$
$$w^{(0)} = \hat{W}^{(0)}(\lambda^n)e^{\xi(\mathbf{y},\lambda^{-1})} \qquad w^{(0)*} = \hat{W}^{(0)*-1}(\lambda^{-n})e^{-\xi(\mathbf{y},\lambda^{-1})}$$

$\xi(\mathbf{x},\lambda) = \sum x_i\lambda^i$, $\xi(\mathbf{y},\lambda^{-1}) = \sum y_i\lambda^{-i}$.

Wave functions satisfy

$$L(w^{(\infty)}) = \lambda w^{(\infty)} \qquad M(w^{(0)}) = \lambda^{-1}w^{(0)}$$
$$\partial_{x_m}w^{(\infty)} = B_m(w^{(\infty)}) \qquad \partial_{x_m}w^{(0)} = B_m(w^{(0)})$$
$$\partial_{y_m}w^{(\infty)} = C_m(w^{(\infty)}) \qquad \partial_{y_m}w^{(0)} = C_m(w^{(0)})$$

Adjoint wave functions have similar expressions

Two dimensional Toda Lattice Hierarchy

Proposition

$$\sum_{k\geq 0} L^k_{\geq 0}\lambda^{-k} = -w^{(\infty)}\Delta_+^{-1}w^{(\infty)*} \qquad \sum_{k\in\mathbb{Z}} M^k_{\geq 0}\lambda^{k} = -w^{(0)}\Delta_+^{-1}w^{(0)*}$$

$$\sum_{k\in\mathbb{Z}} L^k_{<0}\lambda^{-k} = w^{(\infty)}\Delta_-^{-1}w^{(\infty)*} \qquad \sum_{k>0} M^k_{<0}\lambda^{k} = w^{(0)}\Delta_-^{-1}w^{(0)*}$$

where

$$\Delta_+^{-1} = -\sum_{i\geq 0}\Lambda^i \qquad \Delta_-^{-1} = \sum_{i\leq -1}\Lambda^i$$

Proof. Need

$$\mathrm{Res}_\Lambda\, PQ^* = \mathrm{Res}_\lambda\, \lambda^{-1}P(\lambda^n)Q(\lambda^{-n})$$

New extended 2D Toda Lattice Hierarchy

Define $\bar{y}_k$ such that

$$\partial_{\bar{y}_k} = \partial_{y_k} + \sum_{i=1}^{N} \sum_{j \geq 1} \lambda_i^j \partial_{y_j}$$

for distinct non-zero parameters λ_i. Then

$$\frac{\partial}{\partial_{\bar{y}_k}} L = [\bar{C}_k, L] \qquad \frac{\partial}{\partial_{\bar{y}_k}} M = [\bar{C}_k, M] \tag{3a}$$

where

$$\bar{C}_k = C_k + \sum_{i=1}^{N} \sum_{j \geq 1} \lambda_i^j C_j$$

New extended 2D Toda Lattice Hierarchy

Define $\bar{y}_k$ such that

$$\partial_{\bar{y}_k} = \partial_{y_k} + \sum_{i=1}^{N} \sum_{j \geq 1} \lambda_i^j \partial_{y_j}$$

for distinct non-zero parameters λ_i. Then

$$\frac{\partial}{\partial_{\bar{y}_k}} L = [\bar{C}_k, L] \qquad \frac{\partial}{\partial_{\bar{y}_k}} M = [\bar{C}_k, M] \tag{3a}$$

where (according to propositon)

$$\bar{C}_k = C_k + \sum_{i=1}^{N} \sum_{j \geq 1} \lambda_i^j C_j = C_k + \sum_{i=1}^{N} w_i^{(0)} \Delta_-^{-1} w_i^{(0)*}.$$

And $w_i^{(0)(*)} = w^{(0)(*)}(\lambda_i)$ are (adj-)wave functions s.t.

$$\partial_{x_m} w_i^{(0)} = B_m(w_i^{(0)}) \qquad \partial_{x_m} w_i^{(0)*} = -B_m^*(w_i^{(0)*}) \tag{3b}$$

$$\partial_{y_m} w_i^{(0)} = C_m(w_i^{(0)}) \qquad \partial_{y_m} w_i^{(0)*} = -C_m^*(w_i^{(0)*}) \tag{3c}$$

New extended 2D Toda Lattice Hierarchy

ex2dTLH: Use $w_i^{(*)}$ instead of $w_i^{(0)\,(*)}$, (1) and (3) give rise to: When $m \neq k$:

$$B_{m,x_k} - B_{k,x_m} + [B_m, B_k] = 0 \tag{4a}$$
$$C_{m,\bar{y}_k} - \bar{C}_{k,y_m} + [C_m, \bar{C}_k] = 0 \tag{4b}$$
$$B_{m,\bar{y}_k} - \bar{C}_{k,x_m} + [B_m, \bar{C}_k] = 0 \tag{4c}$$
$$B_{k,y_m} - C_{m,x_k} + [B_k, C_m] = 0 \tag{4d}$$
$$w_{i,x_m} = B_m(w_i) \qquad w_{i,y_m} = C_m(w_i) \quad (i = 1, \ldots, N) \tag{4e}$$
$$w^*_{i,x_m} = -B^*_m(w^*_i) \qquad w^*_{i,y_m} = -C^*_m(w^*_i) \tag{4f}$$

When $m = k$:

$$B_{k,\bar{y}_k} - \bar{C}_{k,x_k} + [B_k, \bar{C}_k] = 0 \tag{5a}$$
$$\partial_{x_k} w_i = B_k(w_i), \quad \partial_{x_k} w^*_i = -B^*_k(w^*_i) \quad (i = 1, \ldots, N) \tag{5b}$$

where $\bar{C}_k = C_k + \sum_{i=1}^{N} w_i \Delta_-^{-1} w^*_i$.

New extended 2D Toda Lattice Hierarchy

When $m \neq k$, Lax representation for (4a)-(4d)

$$\psi_{x_m} = B_m(\psi) \quad \psi_{x_k}=B_k(\psi)$$
$$\psi_{y_m} = C_m(\psi) \quad \psi_{\bar{y}_k}=\bar{C}_k(\psi)$$

When $m = k$, Lax representation for (5a)

$$\psi_{x_k} = B_k(\psi) \quad \psi_{\bar{y}_k} = \bar{C}_k(\psi)$$

New extended 2D Toda Lattice Hierarchy

Example (2dTLSCS)

When $m = k = 1$, let $u = u_0$, $v = v_{-1}$, $x = x_1$, $y = \bar{y}_1$

$$B_1 = \Lambda + u, \quad C_1 = v\Lambda^{-1}.$$

Denote $f(n+k) = f^{(k)}$, then

$$u_y = -\Delta(v + \sum w_i w_i^{*(-1)}), \quad v_x = v(u - u^{(-1)}), \tag{6a}$$

$$w_{i,x} = B_1(w_i), \quad w_{i,x}^* = -B_1^*(w_i^*), \quad i = 1, \ldots, N \tag{6b}$$

Under $u = q_x$, $v = \exp(q - q^{(-1)})$, (6) becomes 2dTLSCS

$$q_{xy} = e^{q-q^{(-1)}} - e^{q^{(1)}-q} + \sum(w_i w_i^*)_x,$$

$$w_{i,x} = w_i^{(1)} + q_x w_i, \quad (i = 1, \ldots, N)$$

$$w_{i,x}^* = -w_i^{*(-1)} - q_x w_i^*.$$

New extended 2D Toda Lattice Hierarchy

Another ex2dTLH: Introduce $\bar{x}_k$, such that

$$\partial_{\bar{x}_k} = \partial_{x_k} + \sum_{i=1}^{N} \sum_{j \geq 1} \lambda_i^{-j} \partial_{x_j}$$

When $m \neq k$

$$\begin{aligned}
&B_{m,\bar{x}_k} - \bar{B}_{k,x_m} + [B_m, \bar{B}_k] = 0, \qquad C_{m,\bar{x}_k} - \bar{B}_{k,y_m} + [C_m, \bar{B}_k] = 0, \\
&C_{m,y_k} - C_{k,y_m} + [C_m, C_k] = 0, \qquad B_{m,y_k} - C_{k,x_m} + [B_m, C_k] = 0, \\
&w_{i,y_m} = C_m(w_i), \quad w_{i,x_m} = B_m(w_i), \quad i = 1, \ldots, N, \\
&w^*_{i,y_m} = -C^*_m(w^*_i), \quad w^*_{i,x_m} = -B^*_m(w^*_i).
\end{aligned}$$

when $m = k$

$$\begin{aligned}
&C_{k,\bar{x}_k} - \bar{B}_{k,y_k} + [C_k, \bar{B}_k] = 0, \\
&\partial_{y_k} w_i = C_k(w_i), \quad \partial_{y_k} w^*_i = -C^*_k(w^*_i). \quad i = 1, \ldots, N,
\end{aligned}$$

where

$$\bar{B}_k = B_k - \sum_{i=1}^{N} w_i \Delta_+^{-1} w^*_i .$$

New extended 2D Toda Lattice Hierarchy

Example (2dTLSCS, Hu X.B., Wang H.Y. 2006 [2])

When $m = k = 1$

$$q_{xy} = e^{q-q^{(-1)}} - e^{q^{(1)}-q} + \sum (w_i w_i^*)_y$$

$$w_{i,y} = e^{q-q^{(-1)}} w_i^{(-1)} \quad (i = 1, \ldots, N)$$

$$w_{i,y}^* = -e^{q^{(1)}-q} w_i^{*(1)}$$

This equation is equivalent to the previous 2dTLSCS, under:

$$x \to -y, \quad y \to -x \quad q \to q$$

$$w_i \to -e^{q} w_i^*, \quad w_i^* \to e^{-q} w_i$$

This transformation was discovered by Prof. Hu.

Darboux Transformations (DT) + Method of Variation of Constants (MVC)

Soliton equation with self-consistent sources (SESCS) $\rightarrow$ soliton equation with non-homogeneous terms. This inspired us to use MVC (celebrated in ODE theory) to solve SESCS.

Apply MVC to Binary DT : Grammian det. (Y. Zeng,2001, [3])

- KdVSCS, mKdVSCS, NLSSCS
- Non-auto-Bäcklund BDT: $N \rightarrow N+1$ SCSs

Apply MVC to Ordinary DT : Wronskian or Casoratian det. etc. (R. Lin, X. Liu 2005, [4])

- TLSCS, Ablowitz-Ladik with SCS, 2dTLSCS
- Non-auto-Bäcklund DT: $N \rightarrow N+1$ SCSs

DT and MVC

General Scheme

- Ordinary DT (auto-Bäcklund)
- Eigenfunction $h = f + a \cdot g$
- MVC: $a \rightarrow a(t)$
- Prove: Non-auto-Bäcklund DT: $N \rightarrow N + 1$ sources
- m-iteration of DT (different eigenvalues)

This enables us to construct non-trivial solutions of SESCS from trivial one.

DT and MVC for 2dTLSCS

Recall 2dTLSCS and its Lax representations

$$q_{xy} = e^{q - q^{(-1)}} - e^{q^{(1)} - q} + \sum (w_i w_i^*)_x, \tag{7a}$$

$$w_{i,x} = w_i^{(1)} + q_x w_i, \quad (i = 1, \ldots, N) \tag{7b}$$

$$w_{i,x}^* = -w_i^{*(-1)} - q_x w_i^*. \tag{7c}$$

Under (7b) and (7c), ($u = q_x$, $v = e^{q - q^{(-1)}}$)

$$\psi_x = (\Lambda + u)(\psi), \tag{8a}$$

$$\psi_y = (v\Lambda^{-1} + \sum_{i=1}^{N} w_i \Delta_-^{-1} w_i^*)(\psi), \tag{8b}$$

Darboux Tf and MVC for 2dTLSCS

Proposition. Let h is solution to Lax representation (8). Define

$$\mathcal{D} = \Lambda + \sigma, \qquad \sigma = -h^{(1)}/h.$$

Then Darboux Transformation for 2dTLSCS is

$$\tilde{u} = u^{(1)} + \sigma - \sigma^{(1)}, \tag{9a}$$

$$\tilde{v} = v\sigma/\sigma^{(-1)}, \tag{9b}$$

$$\tilde{\psi} = \mathcal{D}(\psi) = \frac{\mathrm{cas}(h, \psi)}{h}, \tag{9c}$$

$$\tilde{w}_i = \mathcal{D}(w_i) = \frac{\mathrm{cas}(h, w_i)}{h}, \tag{9d}$$

$$\tilde{w}_i^* = \mathcal{D}^{*-1}(w_i^*) = -\frac{\mathrm{S}(hw_i^*)}{h^{(1)}}, \tag{9e}$$

where $\mathrm{S} = \Lambda\Delta_-^{-1}$.

Darboux Tf and MVC for 2dTLSCS

Proposition. Let $h = f + ag$ is solution to Lax representation (8). Define

$$\mathcal{D} = \Lambda + \sigma, \qquad \sigma = -h^{(1)}/h.$$

Then Darboux Transformation for 2dTLSCS is

$$\tilde{u} = u^{(1)} + \sigma - \sigma^{(1)}, \tag{9a}$$

$$\tilde{v} = v\sigma/\sigma^{(-1)}, \tag{9b}$$

$$\tilde{\psi} = \mathcal{D}(\psi) = \frac{\mathrm{cas}(h, \psi)}{h}, \tag{9c}$$

$$\tilde{w}_i = \mathcal{D}(w_i) = \frac{\mathrm{cas}(h, w_i)}{h}, \tag{9d}$$

$$\tilde{w}_i^* = \mathcal{D}^{*\,-1}(w_i^*) = -\frac{\mathrm{S}(h w_i^*)}{h^{(1)}}, \tag{9e}$$

where $\mathrm{S} = \Lambda \Delta_-^{-1}$.

Darboux Tf and MVC for 2dTLSCS

Proposition. Let $h = f + a(y)g$. Define

$$\mathcal{D} = \Lambda + \sigma, \qquad \sigma = -h^{(1)}/h.$$

Then Darboux Transformation for 2dTLSCS is

$$\tilde{u} = u^{(1)} + \sigma - \sigma^{(1)}, \tag{9a}$$

$$\tilde{v} = v\sigma/\sigma^{(-1)}, \tag{9b}$$

$$\tilde{\psi} = \mathcal{D}(\psi) = \frac{\mathrm{cas}(h,\psi)}{h}, \tag{9c}$$

$$\tilde{w}_i = \mathcal{D}(w_i) = \frac{\mathrm{cas}(h,w_i)}{h}, \tag{9d}$$

$$\tilde{w}_i^* = \mathcal{D}^{*-1}(w_i^*) = -\frac{\mathrm{S}(hw_i^*)}{h^{(1)}}, \tag{9e}$$

$$\tilde{w}_{N+1} = a^{-1}\mathcal{D}(f), \tag{9f}$$

$$\tilde{w}_{N+1}^* = \frac{\dot{a}}{h^{(1)}}, \tag{9g}$$

where $\mathrm{S} = \Lambda\Delta_-^{-1}$.

m-time DT

Theorem. Let f_j and g_j ($j = 1, 2, \dots, m$) be solutions to (8).

$$h_j := f_j + a_j(y) g_j.$$

Then

$$u[m] = u^{(m)} + \frac{\widetilde{\mathrm{cas}}^{(1)}(h_1, \cdots, h_m)}{\mathrm{cas}^{(1)}(h_1, \cdots, h_m)} - \frac{\widetilde{\mathrm{cas}}(h_1, \cdots, h_m)}{\mathrm{cas}(h_1, \cdots, h_m)}, \tag{10a}$$

$$v[m] = v \frac{\mathrm{cas}^{(1)}(h_1, \cdots, h_m)\, \mathrm{cas}^{(-1)}(h_1, \cdots, h_m)}{\mathrm{cas}^2(h_1, \cdots, h_m)}, \tag{10b}$$

$$w_i[m] = \frac{\mathrm{cas}(h_1, \cdots, h_m, w_i)}{\mathrm{cas}(h_1, \cdots, h_m)}, \quad i = 1, \dots, N, \tag{10c}$$

$$w_i^*[m] = (-1)^m \frac{\overline{\mathrm{cas}}(h_1, \cdots, h_m, w_i^*)}{\mathrm{cas}^{(1)}(h_1, \cdots, h_m)}, \tag{10d}$$

$$w_{N+j}[m] = a_j^{-1} \frac{\mathrm{cas}(h_1, \cdots, h_m, f_j)}{\mathrm{cas}(h_1, \cdots, h_m)}, \quad j = 1, \dots, m \tag{10e}$$

$$w_{N+j}^*[m] = (-1)^{m-j} \dot{a}_j \frac{\mathrm{cas}^{(1)}(h_1, \cdots, \hat{h}_j, \cdots, h_m)}{\mathrm{cas}^{(1)}(h_1 \cdots, h_m)}, \tag{10f}$$

m-time DT

$$\operatorname{cas}(h_1, \cdots, h_m) = \begin{vmatrix} h_1 & \cdots & h_m \\ \vdots & & \vdots \\ h_1^{(m-1)} & \cdots & h_m^{(m-1)} \end{vmatrix},$$

$$\widetilde{\operatorname{cas}}(h_1, \cdots, h_m) = \begin{vmatrix} h_1 & \cdots & h_m \\ \vdots & & \vdots \\ h_1^{(m-2)} & \cdots & h_m^{(m-2)} \\ h_1^{(m)} & \cdots & h_m^{(m)} \end{vmatrix},$$

$$\overline{\operatorname{cas}}(h_1, \cdots, h_m, f) = \begin{vmatrix} \mathrm{S}(h_1 f) & \cdots & \mathrm{S}(h_m f) \\ h_1^{(1)} & \cdots & h_m^{(1)} \\ \vdots & & \vdots \\ h_1^{(m-1)} & \cdots & h_m^{(m-1)} \end{vmatrix}$$

Solutions

Starting: $q = 1$, $v = 1$, $u = 0$, $N = 0$. Lax representation for 2dTLSCS:

$$\psi_x = \psi^{(1)}, \quad \psi_y = \psi^{(-1)}. \tag{11}$$

Independent solutions to (11) w.r.t. parameter $z = e^{\omega}$ are

$$\begin{aligned} f(n,x,y) &= \exp(n\omega + zx + z^{-1}y), \\ g(n,x,y) &= \exp(-n\omega + z^{-1}x + zy), \\ \frac{\partial^k f}{\partial z^k}, \frac{\partial^k g}{\partial z^k}, &\quad k = 1, 2, \cdots \end{aligned}$$

Different choices will give different solutions.

Solutions

Example (Solitons)

Let $a(y) = e^{\alpha(y)}$, then

$$h = f + a(y)g = 2\exp\Omega \cdot \cosh Z,$$

$$\Omega = \cosh\omega \cdot x + \cosh\omega \cdot y + \alpha/2,$$
$$Z = n\omega + \sinh\omega \cdot x - \sinh\omega \cdot y - \alpha/2.$$

DT + MVC $\Rightarrow$ 1-soliton

$$u[1] = \frac{\cosh(Z+2\omega)}{\cosh(Z+\omega)} - \frac{\cosh(Z+\omega)}{\cosh Z},$$
$$v[1] = \frac{\cosh(Z+\omega)\cosh(Z-\omega)}{\cosh^2 Z},$$
$$w[1] = \frac{\sinh\omega \cdot e^{\Omega}}{\cosh Z}, \quad w^*[1] = \frac{\dot{\alpha} e^{-\Omega}}{2\cosh(Z+\omega)}.$$

Solutions

Example (Solitons)

Let

$$h_i = f_i + a_i(y)g_i = 2\exp\Omega_i \cdot \cosh Z_i,$$

then m-time DT + MVC with different parameters $z = z_i$ $(i = 1, \cdots, m) \Rightarrow$

$$m\text{-soliton}$$

Solutions

Example (Rational Solution)

Let

$$h_k = \frac{\partial^k}{\partial z^k} g + a(y) g.$$

For e.g. $k = 1$, DT+MVC $\Rightarrow$ a rational solution

$$u[1] = -\frac{z}{(\eta + za + 1/2)^2 - 1/4},$$

$$v[1] = 1 - \frac{1}{(\eta + za)^2},$$

$$w[1] = \frac{z^{n+1} e^{\xi}}{\eta + za},$$

$$w^*[1] = \frac{\dot{a} z^{-n} e^{-\xi}}{\eta + za + 1},$$

where $\eta = n + zx - z^{-1} y$, $\xi = zx + z^{-1} y$.

Solutions

Example

Let $h = f + a(y)g$, $h_z = \partial_z f + a(y)\partial_z g$. Then 2-iteration of DT+MVC $\Rightarrow$ a singular solution

$$u[2] = \frac{C_2^{(1)}}{C_1^{(1)}} - \frac{C_2}{C_1}, \quad v[2] = \frac{C_1^{(1)} C_1^{(-1)}}{C_1^2},$$

$$w_1[2] = -2\frac{\sinh^2 \omega e^{\Omega}}{zC_1} \cosh(Z + \omega),$$

$$w_2[2] = \frac{e^{\Omega}\left(D_1^{(1)} \cosh Z + D_1 \cosh(Z + 2\omega) - D_2 \cosh(Z + \omega)\right)}{C_1},$$

$$w_1^*[2] = -\dot{a}\frac{\Omega_z \cosh(Z + \omega) + (Z_z + 1/z)\sinh(Z + \omega)}{2e^{\Omega} C_1^{(1)}},$$

$$w_2^*[2] = \dot{a}\frac{\cosh(Z + \Omega)}{2e^{\Omega} C_1^{(1)}}.$$

Solutions

where

$$C_k = (Z_z + \frac{k}{2z})\sinh(k\omega) + \frac{k}{2z}\sinh(2Z + k\omega), \quad k = 1, 2$$

$$D_k = -(\frac{n}{z} + \frac{k}{2z} + F_z)(\frac{n}{z} + \frac{k}{2z} - G_z)\sinh(k\omega) + \frac{k^2}{4z^2}\sinh(k\omega) + \frac{k\Omega_z}{z}\cosh(k\omega)$$

$$F = zx + z^{-1}y, \quad G = z^{-1}x + zy$$

Conclusions

This extensions applied to many 2+1 systems include

- KP, BKP,CKP,mKP, 2dTL
- Dispersionless KP, q-deformed KP etc.
- all consider symmetry generating vector fields

Apply DT to MVC for solving soliton equations with self-consistent sources

- available for 1+1, 2+1 systems
- provide non-auto-Bäcklund DTs.

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